# COMPLEXITY Section

# A COMBINATORIAL PROBLEM FOR POSSIBLE STATES ON THE ARRIVAL LINE FOR *N* COMPETITOR RUNNERS

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***Abstract***. *In a very small* $\varepsilon$*-time interval, several runners could occupy the same place on the arrival line. This possibility creates a combinatorial problem and a statistical one. The work gives the solution for both problems.*

***Keywords:*** *Arrival line, non-nominal state, nominal state, partial feequency, final frequency, algorithm of arrival line.*

Contents



## 1. Problem formulation. Special notations

A number of $n$ competitors run to reach on the arrival line $S$. Each runner has his special sign (on the shirt): $A, B, C, D$ etc.

Work hipothesis. We denote by $t_A$ the time for the runner $A$ and $t_B$ the time for $B$. If $|t_A - t_B| \in [0, \varepsilon]$, uhere $\varepsilon > 0$ is very small (in

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milli-secomds), then both runners ocupy the same place: Place1= I or Place2 = II or Place3 = III etc.

The total number of runners arriving on a special place is denoted by

| I | II | III | IV etc. |
|---|---|---|---|
| $i$ | $j$ | $k$ | $l \cdots\cdots\cdots p;\ \ i \neq 0,$ where |

$i, j, k, l, \cdots, p \in \{0, 1, 2, 3, 4, \cdots, n\}$ and $i + j + k + l + \cdots + p = n.$

**Definition 1.** The set $(i, j, k, \cdots, p)$ is called **non-nominal state.** By $i$ is known only the total name of runners ocuppying the place I; by $j$ is known only the total name of runners ocuppying the place II etc.

The value $N = N(n)$ is the **total number of non-nominal states**.

*Example 1.* $n = 5,$ (3, 1, 1, 0, 0) is a non-nominal state. The set (3, 1, 0, 1, 0) is not a correctly non-nominal state.

**Proposition 1.** $N = N(n) = 2^{n-1},\ n \geq 1.$ (1)

The variable $s = \overline{1, N}$ counts the non-nominal states.

$n = 4,\ (AB, C, D, -),\ (A, BC, D, -),\ (B, C, AD, -)$ are examples of nominal states.

$T = T(n; s)$ is the total number of nominal states generated by the non-nominal state $s, 1 \leq s \leq N(n).$

$S(n)$ = is the total number of nominal states.

$Place1 = I,\ Place2 = II,\ Place3 = III$ etc. are the positions on the arival line.

$FP(n; s; Name, PlaceX)$ = is the **partial frequency** = the number of favorable cases for a runner to occupy the position $PlaceX$ in the **final classification,** for the non-nominal state $s, 1 \leq s \leq N(n).$

$FF(n; Nume, PlaceX)$ = is the **final frecuency** = the number of favorable cases for a nominated runner (A, B, C, D etc.) to occup the position $PlaceX$ in the final classification.

*Example 2.* $FF(n = 4; A, I),\ FF(n = 5; A, I),\ FF(n = 5; B, II).$

In an informatics model, the values $FP$ and $FP$ could be used as the functions having 4 or 3 variables, respectively.

## 2. Example of non-nominal staes and the total number $N(n) = 2^{n-1}$

Example 3.

| $n=1$ | $n=2$ | $n=3$ | $n=4$ | $n=5$ |
|---|---|---|---|---|
| I | I II | I II III | I II III IV | I II III IV V |
| (1) | (2, 0) | (3, 0, 0) | (4, 0, 0, 0) | (5, 0, 0, 0, 0) |
| $N=1$ | (1, 1) | (2, 1, 0) | (3, 1, 0 0) | (4, 1, 0, 0, 0) |
| | $N=2$ | (1, 2, 0) | (2, 2, 0, 0) | (3, 2, 0, 0, 0) |
| | | (1, 1, 1) | (2, 1, 1, 0) | (3, 1, 1, 0, 0) |
| | | $N=4$ | (1, 3, 0, 0) | (2, 3, 0, 0, 0) |
| | | | (1, 2, 1, 0) | (2, 2, 1, 0, 0) |
| | | | (1, 1, 2, 0) | (2, 1, 2, 0, 0) |
| | | | (1, 1, 1, 1) | (2, 1, 1, 1, 0) |
| | | | $N=8$ | (1, 4, 0, 0, 0) |
| | | | | (1, 3, 1, 0, 0) |
| | | | | (1, 2, 2, 0, 0) |
| | | | | (1, 2, 1, 1, 0) |
| | | | | (1, 1, 3, 0, 0) |
| | | | | (1, 1, 2, 1, 0) |
| | | | | (1, 1, 1, 2, 0) |
| | | | | (1, 1, 1, 1, 1) |
| | | | | $N=16$. |

Remark. At the concrtuction of all non-nominal states one recommends that the string for the position I to be a decreasing string. The same rule is for position II etc.

## 3. The construction of the set of all nominal states on the arrival line. The algorithm of arrival line. The main problem of arrival line

**Definition 2.** Each non-nominal state $s, 1 \le s \le N$ generates the set having $T = T(n;s)$ **nominale states** *m* where each runner is known by its identification sign.

We denote by $S(n)$ **the total number of nominal states** on arrival line *S*. Each runner is known by its identification sign.

There exists the relation $S(n) = \sum_{s=1}^{N} T(n;s)$ (2)

It remains the problem to compute the value $T(n;s)$, for $1 \le s \le N$.

**Proposition 2.** Let $s$ be a non-nominal state $s, 1 \le s \le N$. Then we have the formula (3) illustrated for $n = 4$

$$s, (i, j, k, l),\ i + j + k + l = 4,\ T(n; s) = C_n^i C_{n-i}^j C_{n-i-j}^k C_{n-i-j-k}^l. \qquad (3)$$

The generalisation is immediately.

Example 4. $n = 4,\ s = 5, (i, j, k, l) = (1, 2, 1, 0),\ i + j + k + l = 4.$

$$T(4;5) = C_n^i C_{n-i}^j C_{n-i-j}^k = C_4^1 C_{4-1}^2 C_{4-1-2}^1 = C_4^1 C_3^2 C_1^1 = 4 \cdot 3 \cdot 1 = 12.$$

Example 5. $n = 5,\ s = 13,\ (i, j, k, l, p) = (1, 2, 1, 1, 0),\ i + j + k + l = 5.$

$$T(5;13) = C_n^i C_{n-i}^j C_{n-i-j}^k C_{n-i-j-k}^l = C_5^1 C_{5-1}^2 C_{5-1-2}^1 C_{5-1-2-1}^1 = C_5^1 C_4^2 C_2^1 C_1^1 = 60.$$

Remark. The total number of nominal states $S(n)$ could be calculated by two methods.

Method 1. For a small value on $n$ we construct the whole set and (directly) count the total number $S(n)$.

Method 2. We generate the set having $T = T(n; s)$ non-nominal states and apply the formulas (3) and (2).

**The algorithm of arrival line**

Step 1. Find the total number $N = N(n) = 2^{n-1}$ of al non-nominal states.

Step 2. Construct the **string of non-nominal states** $(i, j, k, \cdots, p)$, for $s = \overline{1, N}$.

Step 3. For each non-nominal state $s, 1 \le s \le N$ we construct **the set of all nominal states.** Each state has $T = T(n; s)$ nominal states, where each runner is known by its identidication sign or number.

Step 4. We save all the natural numbers $T = T(n; s),\ s = \overline{1, N}$.

Step 5. One calculates the total number of nominal states $S(n) = \sum_{s=1}^{N} T(n; s)$.

## 4. Numerical example for $n = 5$

Example 6. For $n = 5$ the runners are $A, B, C, D, E$.

a) Find $N = N(n)$ and construct all non-nominal states.

b) Construct nominal staes and find $T = T(n; s)$, for any $1 \le s \le N$.

c) Find the total number of nominal states $S(n) = \sum_{s=1}^{N} T(n; s)$.

**Soluție.** Aplicăm algoritmul liniei de sosire.

a) $n=5$, $i,j,k,l,p \in \{0,1,2,3,4,5\}$; find the string of non-nominal states.

| | |
|---|---|
| $s=1$ | (5, 0, 0, 0, 0) |
| $s=2$ | (4, 1, 0, 0, 0) |
| $s=3$ | (3, 2, 0, 0, 0) |
| $s=4$ | (3, 1, 1, 0, 0) |
| $s=5$ | (2, 3, 0, 0, 0) |
| $s=6$ | (2, 2, 1, 0, 0) |
| $s=7$ | (2, 1, 2, 0, 0) |
| $s=8$ | (2, 1, 1, 1, 0) |
| $s=9$ | (1, 4, 0, 0, 0) |
| $s=10$ | (1, 3, 1, 0, 0) |
| $s=11$ | (1, 2, 2, 0, 0) |
| $s=12$ | (1, 2, 1, 1, 0) |
| $s=13$ | (1, 1, 3, 0, 0) |
| $s=14$ | (1, 1, 2, 1, 0) |
| $s=15$ | (1, 1, 1, 2, 0) |
| $s=16$ | (1, 1, 1, 1, 1). |

b) Construct the set of nominal states on arrival line, for $n=5$.

| Place1 | Place2 | Place3 | Place4 | Place5 | $(i,j,k,l,p)$; $i+j+k+l+p=5$ |
|---|---|---|---|---|---|
| *ABCDE* | – | – | – | – | $s=1$ (5,0,0,0,0) |
| *ABCD* | *E* | – | – | – | $s=2$ (4,1,0,0,0) |
| *ABCE* | *D* | – | – | – | |
| *ABDE* | *C* | – | – | – | |
| *ACDE* | *B* | – | – | – | |
| *BCDE* | *A* | – | – | – | |

Remark. If we invert the columns Place1 and Place2 we obtain the states of (1,4,0,0,0).

| | | | | | |
|---|---|---|---|---|---|
| *ABC* | *DE* | – | – | – | $s=3$ (3,2,0,0,0) |
| *ABD* | *CE* | – | – | – | |
| *ABE* | *CD* | – | – | – | |
| *ACD* | *BE* | – | – | – | |
| *ACE* | *BD* | – | – | – | |

| | | | | | |
|---|---|---|---|---|---|
| *ADE* | *BC* | – | – | – | |
| *BCD* | *AE* | – | – | – | |
| *BCE* | *AD* | – | – | – | |
| *BDE* | *AC* | – | – | – | |
| *CDE* | *AB* | – | – | – | |

Remark. If we invert the columns Place1 and Place2 we obtain the states of (2,3,0,0,0).

| | | | | | |
|---|---|---|---|---|---|
| *ABC* | *D* | *E* | – | – | $s = 4$ (3,1,1,0,0) |
| *ABC* | *E* | *D* | – | – | |
| *ABD* | *C* | *E* | – | – | |
| *ABD* | *E* | *C* | – | – | |
| *ABE* | *C* | *D* | – | – | |
| *ABE* | *D* | *C* | – | – | |
| *ACD* | *B* | *E* | – | – | |
| *ACD* | *E* | *B* | – | – | |
| *ACE* | *B* | *D* | – | – | |
| *ACE* | *D* | *B* | – | – | |
| *ADE* | *B* | *C* | – | – | |
| *ACE* | *C* | *B* | – | – | |
| *BCD* | *A* | *E* | – | – | |
| *BCD* | *E* | *A* | – | – | |
| *BCE* | *A* | *D* | – | – | |
| *BCE* | *D* | *A* | – | – | |
| *BDE* | *A* | *C* | – | – | |
| *BDE* | *C* | *A* | – | – | |
| *CDE* | *A* | *B* | – | – | |
| *CDE* | *B* | *A* | – | – | |

Remark. If we invert the columns Place1 and Place2 we obtain the states of (1,3,1,0,0).

This artificial method also could be applied for other non-nominal state, in oder to obtain the nominal states.

| | | | | | |
|---|---|---|---|---|---|
| *AB* | *CDE* | – | – | – | $s = 5$ (2,3,0,0,0) |
| *AC* | *BDE* | – | – | – | |
| *AD* | *BCE* | – | – | – | |
| *AE* | *BCD* | – | – | – | |
| *BC* | *ADE* | – | – | – | |
| *BD* | *ACE* | – | – | – | |
| *BE* | *ACD* | – | – | – | |

| | | | | |
|---|---|---|---|---|
| $CD$ | $ABE$ | $-$ | $-$ | $-$ |
| $CE$ | $ABD$ | $-$ | $-$ | $-$ |
| $DE$ | $ABC$ | $-$ | $-$ | $-$ |

$s=6$ $(2,2,1,0,0)$, $(Place1, Placec2, Place3, Place4, Place5)$
$(AB, CD, E,-,-)$, $(AB, CE, D,-,-)$, $(AB, DE, C,-,-)$
$(AC, BD, E,-,-)$, $(AC, BE, D,-,-)$, $(AC, DE, B,-,-)$
$(AD, BC, E,-,-)$, $(AD, BE, C,-,-)$, $(AD, CE, B,-,-)$
$(AE, BC, D,-,-)$, $(AE, BD, C,-,-)$, $(AE, CD, E,-,-)$
$(BC, AD, E,-,-)$, $(BC, AE, D,-,-)$, $(BC, DE, A,-,-)$
$(BD, AC, E,-,-)$, $(BD, AE, C,-,-)$, $(BD, CE, A,-,-)$
$(BE, AC, D,-,-)$, $(BE, AD, C,-,-)$, $(BE, CD, A,-,-)$
$(CD, AB, E,-,-)$, $(CD, AE, B,-,-)$, $(CD, BE, A,-,-)$
$(CE, AB, D,-,-)$, $(CE, AD, B,-,-)$, $(CE, BD, A,-,-)$
$(DE, AC, B,-,-)$, $(DE, AB, C,-,-)$, $(DE, BC, A,-,-)$.

$s=7$ $(2,1,2,0,0)$, $(Place1, Placec2, Place3, Place4, Place5)$
$(AB, E, CD,-,-)$, $(AB, D, CE,-,-)$, $(AB, C, DE,-,-)$
$(AC, E, BD,-,-)$, $(AC, D, BE,-,-)$, $(AC, B, DE,-,-)$
$(AD, E, BC,-,-)$, $(AD, C, BE,-,-)$, $(AD, B, CE,-,-)$
$(AE, D, BC,-,-)$, $(AE, C, BD,-,-)$, $(AE, E, CD,-,-)$
$(BC, E, AD,-,-)$, $(BC, D, AE,-,-)$, $(BC, AE, DE,-,-)$
$(BD, E, AC,-,-)$, $(BD, C, AE,-,-)$, $(BD, A, CE,-,-)$
$(BE, D, AC,-,-)$, $(BE, C, AD,-,-)$, $(BE, AD, CD,-,-)$
$(CD, E, AB,-,-)$, $(CD, B, AE,-,-)$, $(CD, A, BE,-,-)$
$(CE, D, AB,-,-)$, $(CE, B, AD,-,-)$, $(CE, A, BD,-,-)$
$(DE, B, AC,-,-)$, $(DE, C, AB,-,-)$, $(DE, A, BC,-,-)$.

$s=8$ (2, 1, 1, 1, 0). We find 60 nominal states.

$s=9$ $(1,4,0,0,0)$, $(Place1, Placec2, Place3, Place4, Place5)$
$(A, BCDE,-,-,-)$, $(B, ACDE,-,-,-)$, $(C, ABDE,-,-,-)$
$(D, ABCE,-,-,-)$, $(E, ABCD,-,-,-)$ and so on.

There exists a great number of nominal states, namely, $S(n=5)$ is a big number.

c) We compute the value of $S(n)=S(n=5)$, representing the total number of nominal states by two formulas.

We generate the set with $T=T(n;s)$ nominal states and apply the formulas (2) and (3).

$S(n)=\sum_{s=1}^{N} T(n;s),\ N=N(5)=16,$

$n=5,\ (i,j,k,l,p),\ i+j+k+l+p=5.$

$T(n;s)=C_n^i C_{n-i}^j C_{n-i-j}^k C_{n-i-j-k}^l C_{n-i-j-k-l}^p,\ 1\le s\le N$

| | | |
|---|---|---|
| $T(5;1)=C_5^5=1$ | $T(5;2)=C_5^4 C_1^1=5$ | $T(5;3)=10$ |
| $T(5;4)=20$ | $T(5;5)=10$ | $T(5;6)=30$ |
| $T(5;7)=30$ | $T(5;8)=60$ | $T(5;9)=5$ |
| $T(5;10)=20$ | $T(5;11)=60$ | $T(5;12)=20$ |
| $T(5;13)=30$ | $T(5;14)=60$ | $T(5;15)=60$ |
| $T(5;16)=5!=120.$ | | |

$S(5)=\sum_{s=1}^{16} T(5;s)=541$ nominal states.

## 5. The number of favorable cases for a place occupied in final classification. Computation formula for partial frequency

We recall two notions presented in section 1.

$FP(n;s;Name,PlaceX)$ = is the **partial frequency** = the number of favorable cases for a runner to occupy the position $PlaceX$ in the **final classification,** for the non-nominal state $s, 1\le s\le N(n)$.

$FF(n;Nume,PlaceX)$ = is the **final frecuency** = the number of favorable cases for a nominated runner (A, B, C, D etc.) to occup the position $PlaceX$ in the final classification.

Example 7. $FP(n=4;s=3;A,Loc1)$, or better $FP(n=4;s=3;A,I)$,

$FP(n=5;s=3;A,I)$, $FP(n=5;s=5;A,I)$,

$FP(n=5;s=3;B,Loc2)$, or better $FP(n=5;s=3;B,II)$ etc.

There are two methods to compute partial frewuencu $FP(n;s;Name,PlaceX)$.

Method 1. (direct method; direct counting). We generate all nominal states and count their total number, for a nominated runner (*A* or *B* etc.) and a nominated place (I or II etc.).

Metoda 2. (Computation formula for *FP*).

For $Place1$ and the state *s* having the form $(n,0,0,\cdots,0)$ the partial frequency is $FP(n;s;Name,Place1)=C_n^n=1$.

For $Place1$ and, for example, the state $s$ having the form $(i, j, k.l, 0)$ we use the formula $FP(n; s; Name, Place1) = C_{n-1}^{j} C_{n-1-j}^{k} C_{n-1-j-k}^{l}$. (4)

**Proposition 3.** There are many cases to use the computation formulas for $Place2$, $Place3$ etc.

Case 1. We take $T = T(n;s)$ with formula (3) and the state $(i, j, 0.0, 0)$, with $i \neq j$. Then

$FP(n; s; Name, Place2) = T - FP(n; s; Name, Place1)$ (diference's method).

Case 2. We take $T = T(n;s)$ with (3) and the state starea $(i, i, 0.0, 0)$ having two equal values $i = i$. Then

$FP(n; s; Name, Place2) = FP(n; s; Name, Place1)$ (the mathod of equal values).

Case 3. We take $T = T(n;s)$ with formula (3) and the state $(i, j = 3 \cdot i, k = 2 \cdot i, 0, 0)$. Then the computation is based on the proportional method

$FP(n; s; Name, Place2) = \frac{j}{i} \cdot FP(n; s; Name, Place1)$ (proportional method)

$FP(n; s; Name, Place3) = \frac{k}{i} \cdot FP(n; s; Name, Place1)$ (proportional method).

Case 4. We take $T = T(n;s)$ with formula (3) and the state $(i, j, i, 0, 0)$. Then

$FP(n; s; Name, Place3) = FP(n; s; Name, Place1)$ (equal values).

$FP(n; s; Name, LoPlace2) =$

$= T - [\,FP(n; s; Name, Place1) + FP(n; s; Name, Place3)]$.

Now we have to compute the value $FF(n; Name, PlaceX)$ for a nominated runner, with the position $Place$ in final classification.

Example 8. $FF(n = 4; A, I)$, $FF(n = 5; A, I)$, $FF(n = 5; B, II)$.

We put together all partial value $FP$

$FF(n; Name, Place1) = \sum_{s=1}^{N} FP(n; s; Name, Place1)$ (5)

$FF(n; Name, Place2) = \sum_{s=1}^{N} FP(n; s; Name, Place2)$

$FF(n; Name, Place3) = \sum_{s=1}^{N} FP(n; s; Name, Place3)$ etc.

## 6. The complete problem of arrival line

Let $n$ be the number of competitors running to ocuppy a position on arrival line.

Step 1. Find **the total number** $N = N(n) = 2^{n-1}$ for all non-nominal staes.

Step 2. Construct the **string of non-nominal states** $(i, j, k, \cdots, p)$, for $s = 1, N$.

Step 3. For each non-nominal state $s, 1 \le s \le N$ construct **the set of nominal states**.

Step 4. Compute **the partial frequency** $FP$ for Place1, by formula (4).

Step 5. Compute **the partial frequency** $FP$ for Place2, Place2 etc. (proposition 3).

Step 6. Compute **the total number of favorable cases** $FF$ for each runner, in final classification.

Step 7. Arrange (registration) the obtained numerical data $s$, $FP$, $FF$ and $S(n)$ in a **centralisation table** or summary table.

Step 8. Make **the proof of correctitude** for all computations.

Step 9. Attach **the random** variables $X1$ for $Place1 = I$, $X2$ for $Place2 = II$, $X3$ for $Place3 = III$ etc.

Step 10. Make some **statistical computations** for random variables $X1$, $X2$, $X3$ etc.

## 7. The random variables attached to final clasiffication. The probability of each place on arrival line

Let $n$ be the runners $A, B, C, D, E, F$ etc. reaching on arrival line.

From centalization table we know the total number of favorable cases $FF(n; Name, PlaceX)$ for each runner in final classification. We simplify the writing and denote $FF(n; Name, Place1) = FFnI$, $FF(n; Name, Place2) = FFnII$, $FF(n; Name, Place3) = FFnIII$ erc.

The occupied places could be $Place1$, $Place2$, $\cdots$, $Placen$.

**The score rule.** Each runner occupiyng the position $PlaceX$ gives a score (a number of points). The score is at choice of organizors.

For *Place1* the value of score is $\alpha$ points. For example $\alpha = 100$ points.

For *Place2* the value of score is $\alpha / 2$ points or $2\alpha / 3$ points etc.

For *Place3* the value of score is $\alpha / 3$ points or $2\alpha / 3$ points etc.

We define **the discrete random variable** in a classical form (manner) and we take $\alpha = 100$.

$$X = \begin{pmatrix} x_1 & x_2 & x_3 \cdots\cdots x_n \\ p_1 & p_2 & p_3 \cdots\cdots p_n \end{pmatrix}, \ 0 \le p_i \le 1, \ p_1 + p_2 + \cdots p_n = 1$$

$$Xn = \begin{pmatrix} \frac{100}{1} & \frac{100}{2} & \frac{100}{3} \cdots\cdots \frac{100}{n} \\ \frac{FFI}{S(n)} & \frac{FFII}{S(n)} & \frac{FFIII}{S(n)} \cdots \frac{FFn}{S(n\_} \end{pmatrix}, FFI + FFII + \cdots + FFn = S(n) \quad (6)$$

for any natural number $n \ge 1$.

The random variable $Xn$ defines **the random variation of score,** on arrival line. This random variable gives the possibility to execute a lot of statistical compuations [NP] like mean value $M)X) = m,$ variance $D^2(X) = \sigma^2,$ standard deviation $D(X) = \sigma$ non-centered moments $M_k(X),$ centered moments $m_k(X),$ generating function $g(t)$ characteristic function $c(t)$ etc.

Here we mention only tha formulas for mean value and variance $M(X) = x_1 p_1 + x_2 p_2 + \cdots x_n p_n, D^2(X) = M(X^2) - [M(X)]^2.$

## 8. Numerical examples with statistical computations

E**xample 9.** Let $n = 5$ be the number of runners $A, B, C, D, E.$ $N = N(n) = 2^{n-1}.$

a) Compute the partial frequency $FP$ of favorable cases $FP(n; s; Nume, LocX)$ for each runner and any state $s, 1 \le s \le N(n).$

b) Compute the final frequency $FF$ of all favorable cases $FF(n; Nume, LocX)$ for each runner in final classification.

c) Make the correctitude verification.

**Solution.** a) WE find the partial frequency of favorable cases $FP(n; s; Nume, LocX),$ for $n = 5$ runners $A, B, C, D, E.$

Step 1. Constrct al non-nominal states

| | | |
|---|---|---|
| $s=1$ | (5, 0, 0, 0, 0) | |
| $s=2$ | (4, 1, 0, 0, 0) | |
| $s=3$ | (3, 2, 0, 0, 0) | |
| $s=4$ | (3, 1, 1, 0, 0) | |
| $s=5$ | (2, 3, 0, 0, 0) | |
| $s=6$ | (2, 2, 1, 0, 0) | |
| $s=7$ | (2, 1, 2, 0, 0) | |
| $s=8$ | (2, 1, 1, 1, 0) | |
| $s=9$ | (1, 4, 0, 0, 0) | |
| $s=10$ | (1, 3, 1, 0, 0) | |
| $s=11$ | (1, 2, 2, 0, 0) | |
| $s=12$ | (1, 2, 1, 1, 0) | |
| $s=13$ | (1, 1, 3, 0, 0) | |
| $s=14$ | (1, 1, 2, 1, 0) | |
| $s=15$ | (1, 1, 1, 2, 0) | |
| $s=16$ | (1, 1, 1, 1, 1) | $N(5)=2^4=16.$ |

Step 2. For $n \leq 4$ ($n$ with a small value) we concstruct all nominal states on arrival line for given $n$.

Step 3. Compute all partial frequency for Place1, with formula (4).

Method 1. The direct method is difficult to use for $n \geq 5$.

Method 2. Wee use formula (4) for Place1.

$s=1,\ (5,0,0,0,0),\ FP(n=5; s=1; A, I) = C_5^5 = 1.$

$s=2,\ (4,1,0,0,0),\ FP(n=5; s=2; A, I) = C_4^1 = 4.$

$s=3,\ (3,2,0,0,0),\ FP(n=5; s=3; A, I) = C_4^2 = 6.$

$s=4,\ (3,1,1,0,0),\ FP(n=5; s=4; A, I) = C_4^1 C_3^1 = 12.$

$s=5,\ (2,3,0,0,0),\ FP(n=5; s=5; A, I) = C_4^3 = 4.$

$s=6,\ (2,2,1,0,0),\ FP(n=5; s=6; A, I) = C_4^2 C_2^1 = 12.$

$s=7,\ (2,1,2,0,0),\ FP(n=5; s=7; A, I) = C_4^1 C_3^2 = 12.$

$s=8,\ (2,1,1,1,0),\ FP(n=5; s=8; A, I)=C_4^1C_3^1C_2^1=24.$

$s=9,\ (1,4,0,0,0),\ FP(n=5; s=9; A, I)=C_4^4=1.$

$s=10,\ (1,3,1,0,0),\ FP(n=5; s=10; A, I)=C_4^3C_1^1=4.$

$s=11,\ (1,2,2,0,0),\ FP(n=5; s=11; A, I)=C_4^2C_2^2=6.$

$s=12,\ (1,2,1,1,0),\ FP(n=5; s=12; A, I)=C_4^2C_2^1C_1^1=12.$

$s=13,\ (1,1,3,0,0),\ FP(n=5; s=13; A, I)=C_4^1C_3^3=4.$

$s=14,\ (1,1,2,1,0),\ FP(n=5; s=14; A, I)=C_4^1C_3^2C_1^1=12.$

$s=15,\ (1,1,1,2,0),\ FP(n=5; s=15; A, I)=C_4^1C_3^1C_2^2=12.$

$s=15,\ (1,1,1,2,0),\ FP(n=5; s=16; A, I)=C_4^1C_3^1C_2^1C_1^1=24=4!.$

Method 2. Use the proposition 3 for places II, III, IV, V. We obtain the values:

$s=1,\ (5,0,0,0,0),\ T=1$

| | | | | |
|---|---|---|---|---|
| $A1I=1$ | $A1II=0$ | $A1III=0$ | $A1IV=0$ | $A1V=0$ |

$s=2,\ (4,1,0,0,0),\ T=5$

| | | | | |
|---|---|---|---|---|
| $A2I=4$ | $A2II=1$ | $A2III=0$ | $A2IV=0$ | $A2V=0$ |

$s=3,\ (3,2,0,0,0), T=10$

| | | | | |
|---|---|---|---|---|
| $A3I=6$ | $A3II=4$ | $A3III=0$ | $A3IV=0$ | $A3V=0$ |

$s=4,\ (3,1,1,0,0),\ T=20$

| | | | | |
|---|---|---|---|---|
| $A4I=12$ | $A4II=4$ | $A4III=4$ | $A4IV=0$ | $A4V=0$ |

$s=5,\ (2,3,0,0,0),\ T=10$

| | | | | |
|---|---|---|---|---|
| $A5I=4$ | $A5II=6$ | $A5III=0$ | $A5IV=0$ | $A5V=0$ |

$s=6,\ (2,2,1,0,0),\ T=30$

| | | | | |
|---|---|---|---|---|
| $A6I=12$ | $A6II=12$ | $A6III=6$ | $A6IV=0$ | $A6V=0$ |

$s=7,\ (2,1,2,0,0),\ T=30$

| | | | | |
|---|---|---|---|---|
| $A7I=12$ | $A7II=6$ | $A7III=12$ | $A7IV=0$ | $A7V=0$ |

$s=8,\ (2,1,1,1,0),\ T=60$

| | | | | |
|---|---|---|---|---|
| $A8I=24$ | $A8II=12$ | $A8III=12$ | $A8IV=12$ | $A8V=0$ |

$s = 9, (1,4,0,0,0), \ T = 5$

$A9I = 1 \quad A9II = 4 \quad A9III = 0 \quad A9IV = 0 \quad A9V = 0$

$s = 10, \ (1,3,1,0,0), \ T = 20$

$A10I = 1 \quad A10II = 4 \quad A10III = 0 \quad A10IV = 0 \quad A10V = 0$

$s = 11, \ (1,2,2,0,0), \ T = 30$

$A11I = 6 \quad A11II = 12 \quad A11III = 12 \quad A11IV = 0 \quad A11V = 0$

$s = 12, (1,2,1,1,0), \ T = 60$

$A12I = 12 \quad A12II = 24 \quad A12III = 12 \quad A12IV = 12 \quad A12V = 0$

$s = 13, \ (1,1,3,0,0), \ T = 20$

$A13I = 4 \quad A13II = 4 \quad A13III = 12 \quad A13IV = 0 \quad A13V = 0$

$s = 14, \ (1,1,2,1,0), \ T = 60$

$A14I = 12 \quad A14II = 12 \quad A14III = 12 \quad A14IV = 12 \quad A14V = 0$

$s = 15, \ (1,1,1,2,0), \ T = 60$

$A15I = 12 \quad A15II = 12 \quad A15III = 12 \quad A15IV = 24 \quad A15V = 0$

$s = 16, (1,1,1,1,1), \ T = 120$

$A16I = 24 \quad A16II = 24 \quad A16III = 24 \quad A16IV = 24 \quad A16V = 24.$

b) Compute the total number of favorable cases $FF(n; Nume, LocX)$ for any runner in final classification.

$n = 5, \ FF(n = 5; A, I) = 1 + 4 + 6 + \cdots + 12 + 24 = 150;$ similar for $B, C, D, E.$

$n = 5, \ FF(n = 5; A, I) = 1 + 4 + 6 + \cdots + 12 + 24 = 150;$ similar for $B, C, D, E.$

$n = 5, \ FF(n = 5; A, III) = 4 + 6 + \cdots + 12 + 24 = 134;$ similar for $B, C, D, E.$

$n = 5, \ FF(n = 5; A, IV) = 12 + 12 + \cdots + 12 + 24 = 84;$ similar for $B, C, D, E.$

$n = 5, \ FF(n = 5; A, V) = 24;$ similar for $B, C, D, E.$

c) Proof of correctitude.

$$150 + 149 + 130 + 84 + 24 = 541 = S(n = 5). \text{ Correctly.}$$

Wee see the abstract of all results in Table1, for $n = 1, 2, 3, 4, 5.$

**Centralization table 1 for** $n = 1, 2, 3, 4, 5.$

| | *n*=1 | *n*=2 | | *n*=3 | | | *n*=4 | | | | *n*=5 | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | FP | FP | | FP | | | FP | | | | FP | | | | | |
| | I | I | II | I | II | III | I | II | III | IV | I | II | III | IV | V | |
| *s* =1 | 1 | 1 | 0 | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 1 |
| *s*=2 | | 1 | 1 | 2 | 1 | 0 | 3 | 1 | 0 | 0 | 4 | 1 | 0 | 0 | 0 | 2 |
| *s*=3 | | | | 1 | 2 | 0 | 3 | 3 | 0 | 0 | 6 | 4 | 0 | 0 | 0 | 3 |
| *s*=4 | | | | 2 | 2 | 2 | 6 | 3 | 3 | 0 | 12 | 4 | 4 | 0 | 0 | 4 |
| *s*=5 | | | | | | | 1 | 3 | 0 | 0 | 4 | 6 | 0 | 0 | 0 | 5 |
| *s*=6 | | | | | | | 3 | 6 | 3 | 0 | 12 | 12 | 6 | 0 | 0 | 6 |
| *s*=7 | | | | | | | 3 | 3 | 6 | 0 | 12 | 6 | 12 | 0 | 0 | 7 |
| *s*=8 | | | | | | | 6 | 6 | 6 | 6 | 24 | 12 | 12 | 12 | 0 | 8 |
| *s*=9 | | | | | | | | | | | 1 | 4 | 0 | 0 | 0 | 9 |
| *s*=10 | | | | | | | | | | | 4 | 12 | 4 | 0 | 0 | 10 |
| *s*=11 | | | | | | | | | | | 6 | 12 | 12 | 0 | 0 | 11 |
| *s*=12 | | | | | | | | | | | 12 | 24 | 12 | 12 | 0 | 12 |
| *s*=13 | | | | | | | | | | | 4 | 4 | 12 | 0 | 0 | 13 |
| *s*=14 | | | | | | | | | | | 12 | 12 | 24 | 12 | 0 | 14 |
| *s*=15 | | | | | | | | | | | 12 | 12 | 12 | 24 | 0 | 15 |
| *s*=16 | | | | | | | | | | | 24 | 24 | 24 | 24 | 24 | 16 |
| FF | 1 | 2 | 1 | 6 | 5 | 2 | 26 | 25 | 18 | 6 | 150 | 149 | 134 | 84 | 24 | |
| *S*(*n*) | 1 | – | 3 | – | – | 13 | – | – | | 75 | | | | | 541 | |

**Example 10.** Let $n = 5$ be the number of runners $A, B, C, D, E.$

a) Attach the random variable $X5$.

b) Compute the mean value and variance.

**Solution.** a) We use the values from centralization table 1

$N(5) = 16,\ S(n) = S(5) = 541$

$FFI = 150,\ FFII = 149,\ FFIII = 134,\ FFIV = 84,\ FFV = 24$

$$X5 = \begin{pmatrix} \dfrac{100}{1} & \dfrac{100}{2} & \dfrac{100}{3} & \dfrac{100}{4} & \dfrac{100}{5} \\ \dfrac{FFI}{S(n)} & \dfrac{FFII}{S(n)} & \dfrac{FFIII}{S(n)} & \dfrac{FFIV}{S(n)} & \dfrac{FFV}{S(n)} \end{pmatrix}$$

$$X5 = \begin{pmatrix} 100 & 50 & 33{,}3334 & 25 & 20 \\ \dfrac{150}{541} = 0{,}2773 & \dfrac{149}{541} = 0{,}2754 & \dfrac{134}{541} = 0{,}2477 & \dfrac{84}{541} = 0{,}1553 & \dfrac{24}{541} = 0{,}0443 \end{pmatrix}$$

b) Compute the mean value and variance

$M(X5) = 100 \cdot 0{,}2773 + 50 \cdot 0{,}2754 + 33{,}3334 \cdot 0{,}2477 + 25 \cdot 0{,}1553 + 20 \cdot 0{,}0443$

$M(X5) = 54{,}525$ points.

$$(X5)^2 = \begin{pmatrix} 10000 & 2500 & 1111,1155 & 625 & 400 \\ 0,2773 & 0,2754 & 0,2477 & 0,1553 & 0,0443 \end{pmatrix}$$

$D^2(X) = M(X)^2 - [M(X)]^2 = 3851,5058 - 2972,9756 = 878,5302$

$D^2(X5) = 878,5302, D(X5) = 29,64$ points.

Final remark. For $n = 3,4,5,6$ we have the following statistical results.

1) $M(X3) = 70,5067$, $M(X4) = 61,67$, $M(X5) = 54,525$,
$M(X6) = 49,2354$.

If number $n$ is increasing, then the mean value of score is decreasing ($\downarrow$).

2) $D(X3) = 27,8604$, $D(X4) = 28,5303$, $D(X5) = 29,64$, $D(X6) = 29,601$.

If number $n$ is increasing, then the standard deviation of score is increasing ($\uparrow$).

## REFERENCES


[1] Popoviciu Nicolae, *Special chapters of probability and statistics*, Victor Publishing, Hyperion University of Bucharest, ISBN 978-973-1815-98-5, 2014, p. 338.