ECONOPHYSICS Section

QUID OF THE STATUS OF THE TIME FOR CREATIVE MANAGEMENT? A THEORETICAL APPROACH THROUGH ZETA RIEMANN FUNCTION

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Abstract: *Still young, the Professor J. Tenreiro Machado, passed away in Porto October 6, 2021. His deep scientific work had mainly concerned by the nexus between entropy, arrow of time and fractional operators. Our own works confirm the importance of the selinks by introducing how they operate through zeta Riemann function and foliation of dynamical fractional sites. Analysis of fractional Fourier spectra highlights the deficiencies of the usual notion of time and the need for its replacement by sheaves parametrization above dual dynamical Grothendieck's topos. This conclusion is based on the partition of this topos into a couple of tensorial monads whose Cantor's metrics are« $\alpha<1$ » and « $1-\alpha<1$ » (non-integer dimension and co-dimension of fractal geometries folding "spectral geodesics"). The vicinity analysis of each possible state leads a pointed torus topology associated to dynamical multi-connectivity. The authors analyze the existence of the stacking of the universal spectra as origin of Riemann zeta function. This stacking accounts for arithmetic scaling that, depending on α value can play or not a role analogous to a Newtonian time. The authors show that, with the exception of cases $\alpha=1$ and $\alpha=1/2$ (Riemann hypothesis), the stacking parameter used to calibrate the proper time of the system, is different of the zeta complex parameter and then equipped by an" arrow of time" related to a constraint of order at infinity.*

Keywords: *fractional calculus; fractional derivative; Riemann function; entropy; category.*

1. Introduction

This note is a scientific variation which originates from the remark of Henri Poincaré, which I discussed with my friend J. Tenreiro Machado,

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especially during our last meeting in Kazan 2018. Henri Poincaré was dubious about the fact that quantum mechanics uses absolute Newton'stime to develop essentially its discrete concepts. It is obvious that the laws of quantum mechanics do not change by modifying the origin of the Space-Time, thus in accordance with Noether's principles, the role of the symmetry of time and the role of energy as the fundamental quadratic invariant was never contested. However, the relevance of Poincaré's remark has never been challenged. The spectral discretization of the problems; the existence of constants such as that of Planck or that of Boltzmann; the role of probability measures on complex field; the problems of entanglements or questions of finitary constraint, exhibit still open epistemological issues. In particular, despite the paradigms at stake, energy can in no way be an invariant of systems equipped with a memory, as fractional systems are. Open questionsare for instance the following: what kind of invariant can replace the energy when fractional complex systems behaves for example recursively up to infinity? By what mechanisms can complex systems lead to "universal singularities" as introduced by Daniel Sibony [1] ?

The purpose of the current synthesis is focused on a critical approach of temporality seen as an inverse Fourier transform of frequencies spectra. In discrete complex systems, the cardinal concept of entropy appears as having precedence over the notion energy. To reach this conclusion, the authors will study the fractional dynamics. By proving to be dual and bringing multi-connectivity properties, these dynamics will question the link between the non-integer metric and the topology of the systems and incidentally its entropy. In doing so, the analysis will give a particular perspective to Carlo Rovelli's point of view [2] according to which a change *per se* does not exist anymore than the notion of absolute time, these concepts only making sense technically and relatively to each other. This thermodynamical generalization of relativity will be examined here by using self-similarity, namely a non-integer metric, to shift the differential approach to a monadic folding one. The central question will then be reduced to the conception of a non-standard canonical and dual temporality bringing us closer to the perception of the consciousness with a time which flows irreversibly like "*anot her kind of time*" [1] in tensorial links with the Newton one.

At least since Galileo, the practice wants that a motion (if the physical extensity is space (L)) and a variation (if the extensity at stake is other than space although often proportional to L) are represented by a dynamic process using a single automorphism parameter called the *time*. To do so scientific methods use a quantity called velocity (Lt^{-1}). The mathematical content of this notion for any continuous function is a limit named "derivative". Aside, it is showed that if the physical processes are independent of the time origin, then the dynamic processes are settled by an invariant called Energy, namely a quadratic form of a certain velocity. All formal mechanics (Hamiltonian, Lagrangian in particular) are based on Noether's principles. Implicitly this suggests that, even discrete, the processes can be fitted onto continuous and differentiable templates. But what about when they are not differentiable, as in fractional dynamics?[1]. Fortunately, around 1850 after Kummer and Dedekind studies Emmie Noetherhas developed the notion of ideal opening the way to the notion of *filter* and of *ultrafilters* later conceived by H. Cartan in 1938to remove many limitations about systems characterized by partial orders and differentiability issues. These concepts will be used below at least implicitly, in a frame of a categorical approach of time irreversibility.

2. Energy, entropy and time symmetries

Even now some scientists still situate thermodynamic formalism at the epistemological opposite of the mechanical formalism. Respectively linked to an (operative) ingenium *versus* a (contemplative) cosmology the philosophical approach seems obviously different. Nevertheless, this initial opposition between both sciences partly vanished with the design of statistical thermodynamics and then of statistical mechanics. However, as Cédric Villani's works point out, the constraint related to the hypothesis of continuous distributions mainly differentiable for any macroscopic behavior while the microscopic behavior is discrete (hypothesis of hard or soft balls collisions), involves huge mathematical problems that are still unresolved [3]. These difficulties are mainly due to the writing of problems within the framework of set theory under the ZFC axioms. For addressing thermodynamics of irreversible open systems likely to be non-additive, and self-similar (or monadics), we suggest here to oppose to the Maxwellian vision of time, a categorical approach, using *dynamical sites based ons heaves above topos*. Such systems should highlight the fact that if energy

[1] Contrary to the PDE approach, it will be observed that non-integer derivation operators, being operators with memory, the yassign values other than zero to the derivative of the constants. The arbitrary calibration of these properties (derived from Caputo and others) certainly makes it possible to deal with practical problems of control of fractional systems but does not directly address the epistemological consequences of kind of characteristic, which we propose to face here.

stays an invariant of the physics, it is most likely a dual quantity each component having no direct relation with energy but mainly with entropy. Dual on the one hand because any "*quantity*" is always based on the product of an Extensive quantity by an Intensive quality [4] and dual on the other hand because the energy involved in a process must be written as the sum (coproduct) of an efficient quantity (free energy) and a heat loss associated with an irreversibility of the process (entropy). This loss expresses the resistance to a change and is expressed by the product of a temperature (intensity as cardinal) and an ordinal function of a combinatorial nature called entropy (here a flow of extensities). Any finite physical process is characterized by a thermal effect (irreversibility) and the associated loss of efficiency increases with the power involved (energy in stake per unit of time). For some processes the inversion of the flow of extensities makes it possible to find the initial state (example charge of a battery or loop of catalysis) but this always singular operation is difficult and expensive because during any natural transformation (in the physical and not categorical sense) the entropy naturally increases. Entropy is the only oriented function in the physics of change. This function seems oriented as an arithmetic count can be, for example in the monoid (**N**,+): $n \rightarrow n + 1$ for instance represented by an oriented straight line. Far from the Newtonian time of the standard mechanics, entropy is too naturally assimilated to the irreversible time flowing over a geodesic likely to be reduced to this monoidal straight line.

The ambiguity concerning the reversibility of "mechanical time" *versus* the irreversibility of "thermodynamic time" is weaved by the following fact: since any change can only be featured with respect to an temporality external to the system *per se* (the beating of 'a pendulum, the vibration of a molecule, the electronic transition of an atom) the hypothesis according to which time itself is subject to change requires that the temporal frame of reference be defined in relation to which this change must be considered. The only possible solution in the state of knowledge may seem highly paradoxical: "*any arrow of time must be evaluated in relation to time itself*". This statement is inconceivable without a duplication of time to set an internal standard of the dual. Therefore, if time is counting down, this phenomenon can only be related to a duplication of the set **N** itself. Would this property not be linked to the quadratic self-similarity relationship that equips the set **N** namely **N**=**N**x**N** and then the irreversibility perceived would not be related to self-similar metrics, our way of counting and measuring? How would this be thinkable without calling for the use of a recursion operator as it appears, for example, to design monadic categories [4]? Evoking the “arrow of time” unrelated to a process involving “self-referencing” seems at this stage to have no heuristic future. Even without support of the self-similarity, evoking the irreversibility of time *per se* seems to be a metaphysical way of twisting a problem that remains, at this step, conceptually vague. However, in accordance to the fact that a pendulum always merges in a linear way space and time, it is possible to leave the metaphysical field by pressing the concept of time on a non-integer space metric. Beyond the idea according to which time is only defined by means of a pendulum imposing a "granularity" to measures and moreover the use of complex field, the technique defining the device for measuring time, underwrites that, like in Foucault experiment, irreversibility (angle shift) or reversibility (no shift) is related to the referential considered [3]: respectively the whole mass of the universe (inertial Galilean) or only our planet (not inertial). A correlation is therefore established between a measure of a change in a given unit of time without absolute reference and a variation after counting of numerous oscillations. These correlations will almost surely be explained by a non-linear relationship between granular time and extend measure brought at stake by any kind of energy transfer generating entropy; among these extensities requiring a measure, the space which is at the foundation of any metric. This hypothesis leads to assign a granular character to the flow of physical time and with its projection over ideal basis, thus its spectral character. Beyond, finitary rings properties and self-similarity metrics will naturally introduce a monadic generalization of ring properties. The spectral point of view amounts to giving pre-eminence the distributions over functions and duration over momenta. The test functions always required for measuring, namely the distribution theory, will also be used to unfold the ambiguities and reveal hidden symmetries. The size of the clock dial, like the length of the pendulum then associate the temporal issues previously defined with spatial quantities. *Space-Time* as the only dual entity, then enters in the cognitive game development, except that the non-linear character of the coupling introduced for instance by the recursiveness playing between cardinal and ordinal data, cannot be satisfied with the simplification given by a simple dial(complex plane) or a simple wire or rod for pendulum. The dial, like a camembert, must be cut and breach at least into a couple of pieces for dual coupling to appear and cause the emergence of morphisms, namely correlations, symmetries and then geometrical entropy. These statement raises immediately, among other things, the question of the divisibility and the morphism of dials, as scanning approach of the durations and memories. A number "*a*" is divisible by a number "*b*" if there is a number "*k*" such that: $a=k.b$. Divisibility therefore uses the monoid (**N**,x). The existence of a cut of the plane then requires the use of a complex field formalism involving the reference $I=(-1)^{1/2}$. This statement may seem paradoxical since unlike usual time, the field of complex numbers does not offer a natural order, but, because any change in a system may be referred to an external clock, the binarity of complex plane is nonetheless a powerful structure that deserves of being explored for understanding the deepness of the time concept.

3. Statistics and/or fuzzy dance?

Usually the temporality is represented by a standard oriented continuous real line and thus we can obviously associate a flow of a discrete module of time to the arithmetic successor operator in the monoid (**N**,+) namely $n \rightarrow n+1$. But, in what context relating *continuous* to *discrete* is the unit of time really comparable to this too obvious unfolding operator? Zeta function takes place exactly at this gloomy node as a **N**-test of continuity within discrete sets. The major property of **N** is to be a set equipped with a good order which allows to have a principle of induction up to countable infinity. This principle is equivalent to the statement that every subset, namely all parts of P(**N**), have a smallest element. From a categorical point of view, this means that we have the notion of lower limit but not natural upper limit. The asymmetry pointed out is here strictly axiomatic and linked to the successor operator in **N**. The indifferent nature of the orientation of time, that is a characteristic of the mechanical laws cannot then simply be associated with acounting oriented by the succession operator. It can only be so if the physical laws are indifferent (or not) with regard to the "sign" of the extensity in stake when counting beat of time (for example symmetry of space for the clockwise or anti-clockwise or asymmetry of the electric charge, for example for the positron which “goes back in time” in Dirac models etc.). In practice, if the mathematician affirms as an axiom the existence of the successor operator, the physicist can in no way be satisfied with a transposition of the orientation of counting in order to affirm that irreversibility is present. The succession in the monoid (**N**,+) is therefore a naive point of view that cannot be transposed without care to the definition of the notion of physical time. However, given the above, what about the use of its dual multiplicative monoid (**N**, x) in arithmetic? Could it not have much greater advantages by taking into account the set of rational numbers **Q** involving an internal order much more subtle than that of **N**? Would the scaling associated to division not capable to model the concept of formal irreversible time? The possible choice of the set **Q** may seem paradoxical because it operates in the absence of a lower limit of the subsets. It could however be relevant since on the one hand this absence is likely to generate a *boundary uncertainty* (uncertainty on initial conditions then naturally opposite to Noether's principle) and that on the other hand, due to intensive possible meaning, **Q**, resulting from (**N**, x) can naturally take into account the notion of *duration*, the notion of *moment* being then reserved **Q** *et* $n \in$ **N**. The interest of this inversion of the natural order of **N** reinforces the Carlo Rovelli point of view who asserts, after a detailed examination of the problems of mechanics, that a universal absolute time does not exist and that the examination of relative changes, *namely a categorical adjunction*, is sufficient to represent any change. This epistemological position is implicitly based on the notion of *functor* and *Natural Transform*, always coupled subsets being concerned by *recursive morphisms* [5,6]. However, this principle based on relativity for any changing implies that any point of view on evolutions be at least dual, equipped of a quantifier, and therefore inscribed within the framework of a *categorical ordering* which questions the identity of the categories and the logics in stake [7-9]. One of the components of the duality associated can be for example a clock providing a standard beat. Nevertheless, any dual point of view using a norm, can be reversed, for example the same pendulum becoming object of interrogations since any measure requires vice-versa the presence of a co-data. We will see from the very particular case of the inversion of the monoid (**N**,+**)** as a normalized straight line, the potential relevance of the approach backed by the monoid (**N**, x) as normalized geodesic. Before that, let us observe, following Maxwell and Boltzmann works, that the link between the mechanical sciences and the thermodynamic sciences are mainly based on statistics, or even on a theory of strictly additive and normed measures namely very special and among the simplest measure theory based implicitly on Logarithmic function (see at the opposite even in mechanics the information sciences, as shown by Taganov and Babenko [10]) Considering the behavior of ideal gases, Boltzmann was led to construct a –H function expressing the statistical homogenization of the distribution of velocities of the molecules of a gas, initially in an inhomogeneous state. The natural evolution of this H-function (called relaxation) is similar, up to the sign, to that of the entropy function S based on logarithm. However, the irreversible orientation of this type of relaxation is based on perfect reversibility of the local elastic shocks (as reversible as the shock between balls on a pool table). The reversibility of the “atomic” shock (local) versus the irreversibility of the set of shocks (global) at long time then appears a paradox always open [11,12]. This leads to the impossibility of reversing the temporal order of the macroscopic process, which then appears as linked to a change of scales of analysis (local *versus* global). As shown using Maxwell's demon "thought experiment", this link seems to be assignable to an accumulation of irreducible not countable uncertainties, in fact related to the definition of the subsets, in the set of information at disposal. Therefore, information has to be considered statistically. The oriented H-function imposes a point of view originating in the details followed by integration (i.e., a bottom-up series). However, in light of what category theory [6] teach us about counter-intuitive and dual ways of thinking, fractal geometries, (namely the whole is in the part; the whole is more than the sum of the details; capacitive measures can be non-additive; completion is not necessarily acquired and requires the duality of an embedding space etc. (11)) would it not be possible to geometrically circumvent standard set statistics, – seen through measure theory (additive and normed measures) –, for switching the analysis according to a categorical and more precisely monadic and recursive approaches ignoring the usual differential way of thinking? From then on, Boltzmann's approach could be generalized through new non-integer space-time, the scale changes then leading to consider non-additive measures, whose normalization and closure would constitute degrees of freedom. The scaling then would be seen as a sequence of filters (converging series of gauges and templates) in geometries which for simplicity could be self-similar? The transposition from the differential point of view to the monadic point of view may appear clearly disruptive. Far beyond the trivial intuition linked to a simple succession, it highlights, a strong relationship with scaling hidden in its projective arithmetic. Thus, the ontological question of a dual temporality should be then open [13].

As Prigogine stated [14], not without having been roundly and paradoxically criticized by experts in formal quantum mechanics, irreversibility of processes is obviously not related to our measurements; they are intrinsically irreversible and the “presence” of external observers and therefore of measurement uncertainties is, in no way, the cause of an irreversibility that would orient or order our consciousness of time. If irreversibility is linked to statistics operating on locally reversible processes, irreversibility cannot be linked to local measurements, but to the taking into account of "global characteristics" associated with scale changes (namely the set of subsets) that can go beyond the usual statistical framework. As it is involved by the spectral series, any physical change infers, a passage from the local to the global, and therefore a role of the metric (monadic aspect replacing the one parameter self-morphism) and of the topology (through vicinity). We find here a well-known cosmological distinction between the notion of curvature or parallel transportation (via the measurement of metric herein linked to the local mass density) and the notion of topology namely vicinity and connexity associated with principles of least actions. By taking into consideration the asymmetrical union of a meta-categorical nature $((A \cup B) \cup (A \cap B))$which underlies all self-similarity and dualizes the operation of succession, intuition then discreetly distances itself from the notion of arithmetic succession, to take into account a monadic kernel $(T^2 \sim T)$and a deployment through a Keisli algebra. Although still very weakly, one perceives here the emergence of an epistemological turning point supported by a recursion and fractal geometry both suggesting a progression that is no longer linear but looped and spiral, the "proper time" of the dynamic system being associated with a succession but in the order of scale therefore related to (**N**,x) and no longer calibrated according ergodically to "a standard distance" namely to (**N**,+) [14].

Importance of iterated operators, as a monadic generalization of successor operator in a spiral ring is not a detail in Ilia Prigogine's works [15] who explicitly links entropy to a composite recursive process illustrated by means of the transformation of the baker. This transformation associating 2D (composition) and 3D (associativity) -*folding* 3D*, spreading* 2D*, folding* 3D*, spreading* 2D*, etc.*-, leads very subtly to distinguish stages of dual effectiveness leading to two neighbor points at a step $2n$, find themselves very distant (or very close) at step $2(n+\omega\tau)$, in accordance or not with the Liapounov divergence laws [14]. It is established that the baker transformation is strictly irreversible; there is indeed no reverse process allowing to find the initial state because it is not only a question of considering two points[2] which diverge but of taking into account all the possible subsets of points of the baker dough seen as the continuity of an infinity number of subsets. The problem of irreversibility is well and truly here a topological problem of distance, connexity and vicinity of subset [11,12,14].

[2] The problem raised here is of a topological nature because it is questions of vicinities that are at stake. A physical object cannot be thought of in itself, it must be thought of in its categorical context by addressing open questions (Yoneda's lemma). But changing vicinity in a given topology amounts to change scale therefore to consider filters (H. Cartan), the entanglement of the latter giving rise to grids (G. Choquet).

Associate time with the variable $m = \omega\tau$ as it would seem natural does not hold a *disputatio*, because the fact of admitting that time constant τ of the composite operation of folding (2D) and spreading (3D), is unique and irreducible and therefore represents a "statutory and external arrow of time", leads to a tautology and thus proves nothing with respect to the "internal field of velocities" of the dough. By choosing an arbitrary standard length, each clock time can then be associated with a distribution of times characteristics erecting a local sheaf-times. Notwithstanding the definition of a repetition time "τ" and that the occurrence "2τ" is the same transform in the frame of monadic properties, the baker's transformation can be associated with an increase in entropy, -i.e. an increase in the internal disorder of the dough and entanglement of the set of points initially neighbors-, only by taking into account the crisscross of sheaves (partly based on the recursive algebra involved by the monadic structure). The resulting ambiguities (Galois denomination) in terms of internal past-present and future, is associated with the development of a multi-connectivity which designs the final topology of very complex definition of what vicinity means. The recursion of the dynamics then involves long range correlations of the time-sheaf (spectral memory and prophecy) with the most often emergence of scaling properties optimizing the density of entropy-flow within the dough.

Arrived at this stage of our questioning the heuristic progress of our arguing seems very small because if the entropy is an extensive function clearly defined and oriented at the combinatorial level with as an emblematic example the Shannon formulation $S = -\sum [p_i \, log(p_i)]$ at the opposite, the being of time in its consciousness complexity related to sheaves of relationships remains at this step a categorical open question, as well its links with the determinism of the successor and divisor operators respectively in additive and multiplicative monoids. The relationship between "time-sheaves" and causality is equally fully open. However, it is at this stage, by critically examining the paths previously sketched out, that we can make the hoped-for conceptual leap. To do so, we propose to revisit three questions:

- The mathematical meaning of the logarithm function and the implications of its extensive use in physics.
- The use of Fourier (or Laplace) transforms as time transforms and scaling measurement operator and its generalization.
- The relation between self-similarity and renormalization (the recursive geometries), associated with the spectral concept of time.

4. Logarithm & Exponential functions versus asymmetrical bi-partitions

Any sassy student can only be surprised at the extensive use of the logarithm and the exponential function in science and in engineering. In the framework of category theory, this couple of functions expresses a duality: namely the duality of the product (projection) and the co-product (sum). It is clear that their unbridled use first and rightly points to arithmetic constraints much more than physical ones. Any physical measurement passes implicitly or explicitly through the use of the monoids (**N**,+) and (**N**, x), and thus through their links, or even for scaling reasons through the use of ring with modules and field. Obviously, any measurement requires the algebraic links between addition (subtraction/0) and **Z** field and multiplication (division/1) and **Q** field. Any computation calls out the serial character of composition (2D) while associativity plays with diagram division (3D). In this matter, the logarithm (log) function has a very special status. Indeed, we call logarithm the aera $A(x)$ ($A=-log(x)$) that below the hyperbole $f(x)=1/x$, is located between the abscises 1 and $1/x$, namely a kind of integration which is expressed otherwise by writing $dA/dx=f(x)=1/x$, or after permutation of the variables :$x\sim exp[A(x)]$. Therefore, the *logarithm* function qualifies numerically the homogeneous set of subsets of sizes larger than $1/x$[3]. *The log function is thus explicitly attached to a problem of bounded and linearized scaling (by using a sum) from an initial state to a minimum size of resolution stated by* $1/x$. In this perspective, writing informational entropy likea sum of logarithms ($S=-\sum [p_i\ log(p_i)]$) is nothing but a tautology: its orientation during the deployment of natural relaxation processes designates a change of scale for which it is not equivalent to go in the direction of small scales towards local detail (top down) or to go in the direction of the large scales according to a synthetic approach namely the "coarse graining" (bottom-up). This logarithmic approach is nothing else than the simplest "Matryoshka approach". There is a stacking order of the Russian dolls although they are all strictly identical except the linear scale in between. The freedom to compute either using sums (of amalgamated details) or using product and projection over an "ideals" basis as Noether suggested, is expressed in the duality log/exp, *however this duality conceals a constraint of homogeneity and implicit simple kind of memory, most often*

[3] If x is a prime number, then the generalized sum is related to the concept of filter and more over ultrafilter.

passed over in physics lectures. The derivative $1/x$ supposes a homogeneous division of the whole set X. We take an object X and cut it into x objects that are totally *indistinguishable* from each other. The set theory under ZFC axioms is then tacitly involved in every homogeneous point of view, therefore also the standard Peano-arithmetic. Each part is an atomic point and behaves like a wheel among others identical [11].

The critical analysis of this homogeneity can leave the standard approach precisely at this step. Conversely to a homogeneous fragmentation leading to indiscernibility of parts, a non-homogeneous splitting, assigning different relative "weight supports" to a generic inhomogeneous bi-partitio[4], namely a *distinguo* between α and $1-\alpha$, immediately invalidates the use of standard set theory because Cantor sets are then involved through iteration and hence a thoughtless use of the logarithm function becomes cognitively risky; likewise the use of the standard entropy which is nothing else that a simplest functional variation about homogeneity and indistinguishability of sets in ZFC theory. With the introduction of the parameter α which is a metric one, the meaning of the variable x changes from that of a x-partition (cardinal) to that of a scaling variable (ordinal). We will see later how the logarithm function nevertheless regains some of its relevance in the particular case of dual power laws, but let us first examine what are the consequences of this metrical *distinguo* on the exponential function.

To connect the following reasoning to the above an give a metrical meaning at α, let us observe that the evoked inhomogeneity is, in subsets point of view, naturally associated with the notion of identity, iteration and recursion. Assumption of inhomogeneous partition based on α and $1-\alpha$, immediately brings to mind the construction of a Cantor set, therefore the notions of non-integer dimension and fractal co-dimensions. Indeed, by inversion of standard fractal Mandelbrot formula, the fractional metric formula for Cantor sets is such that $1 \sim n.\,\eta^{\alpha}$ if n is the number of pieces of

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[4] The parameter α quantifies a relative weight of the inhomogeneous partition. This is an analytic division operation and therefore a basic engineering operation. We will draw from it a notion of cantor dimension, obviously not to be confused with multiple other definitions of the dimension which, must be stable in a given context, for example metric (distance), topological (with the notion of connectedness), algebraic in the sense usual or homological etc.

the hollow X-puzzle and $\eta=1/k$ the average scale of each pieces 5, in other words: $\alpha.log(\eta) + log(n) = 0$. Iteration when associated with a generic in homogeneity leads at limit to a fractal structure [11]. It erects a set of weight 1over a set of zero measure, immersed in a set of measure 1 with a zero weight. In physics the weight is given through a dynamic of exchange: TEISI non completed model [16,17]. Above two dual sets have respectively for fractional dimensions precisely the values α and $1-\alpha$.6 α is the dimension of the cantor set and $1-\alpha$ is the dual co-dimension of void and then the dynamics can be expressed using fractional differential operators [16-18]. The critical case is obviously $\alpha=1/2$ since the fractional co-dimension is then strictly equal to the dimension and the immersion becomes a self-immersion (Peano curves with $d=1/\alpha$). Using simple duality, let us observe that we can give to all this quadratic problematic a content of Peano metric and no longer of Cantor one; indeed: $1=\eta^2 xn$[7]. Therefore, $n^2+m^2=k^2$ namely $1/\eta_n+1/\eta_m=1/\eta_k$ fulfills Pythagorean relationship. The "external" world, namely the co-universe of immersion, disappears from our intuition because it is certainly a-causal but similar to the causal one. Let us observe that there is then an obvious link between the Pythagorean relation and this fractional metric since the respect of the quadratic relation for a sum imposes the usual diffusive correlation characteristic $D_{corr}=L^2t^{-1}$. The "time" variable t is here nothing else than the inverse of a frequency p of weighted correlations, namely a scale distribution which can then be thought of in the field of complex numbers through a spectral approach of fractal dynamics and Cantorian representation [16,17].

Still using the concept of duality, the problem opened by the inhomogeneous α and $1-\alpha$ bi-partition and then iterations, must also be expressed via the exponential function but now in an infinitely more subtle way than for the linear logarithmic form $\alpha log(x)$ and $(1-\alpha)log(1-x)$. It is

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[5] Let us observe that for $\alpha=1/2$ we will affirm that the self-similarity is quadratic because this metric is related to the 1D Peano curves which have the property of covering a (2D) plane fulfilling in particular the complex plane.

[6] It will be observed that the base of the logarithm becomes one more degree of freedom for our representations, that this degree also opens up a problem of change of scale arising the tropical geometry which is therefore asymptotically linked to our problem.

[7] Let us observe *i* that in the general case of inhomogeneity of the recursive division the questions of non-additivity of magnitudes and non-linearities of distributions arise exactly in the terms in which they arise in fractal geometry and in the theory of capacitive Choquet measures.

no longer a question of starting from the piece of the puzzle to arrive at a "hollowed out set", but of starting from the "hollowed out universe" to express the possible distribution of the sizes of the pieces fitting together to represent the set at all scales of the puzzle. In practice, it is a matter of looking for a canonical basis for the distribution of relevant templates allowing the puzzle to be projected onto *ideal orthogonal subsets*[8]. With declination from the homogeneous case $x=exp(-kA)$ which is, at this step of generalization, our only tool of thought, we must imagine all the conceivable possible divisions arithmetically consistent at the various scales with the recursion of the bi-partition α and $1-\alpha$. For this, we need to go through a spectral analysis. This analysis is very simple if homogeneity is acquired. Indeed, the scaling relation, here linear, is the following $1/\tau = n.\eta$ because by writing $t/\tau \sim A$, then the Fourier Transform [TF(n)] is a TF of an exponential which must be written like a transfer impedance $Z \sim 1/1+p$, with only one $p=i\omega\tau$ as *TL* (t/τ). Passing to the Fourier Transform (TF) the passage to the field of complexes is reduced to an unfolding of scales along the complex axe *"i"* with $p=i\omega\tau$ with a only norm τ. There is certainly an infinity of workable partitions but all of these refer to a single normalization of the exponential transfer function: $x=exp-t/\tau)$. Although elementary, this analysis of the linear link between scaling factor through $A=-log\ n$ and *time* makes it possible to bring out new trails which, to our knowledge, have not been explored by any other authors, at least so simply:

- the first is that even the traditional approach to time is in agreement with a time t no longer considered as a natural and intuitive notion of shift factor, but as a subsidiary notion as an inverse spectral transformation, TL^{-1} (TF^{-1}) of the variable of Laplace p (respectively of Fourier). The spectral statement acquires a physical meaning with the consideration of a scaling (as produced by p (if derivative) or as a divisor $1/p$ (if integrator)). The central notion of the dynamic would therefore no longer be t but p. As soon as a scaling becomes nodal, we have to think in a spectral way, therefore in the fonctorial space of recursive sequences, in magnitude (spreading) and in series (backing). The "*spectral structure*" then replaces the "*space-time*" which we use out of habit to build our "*objective and holistic*" representations of

[8] Implicitly we reformulate here mathematical questions known under the qualifier of cyclotomics.

unity via like in puzzle, scaling for change. Virtually, both FT and TL explore the different scales of the structure. It is through the *p*-scaling (also scanning) that the usual concept of temporality is conveyed in and through scientific experiments or even made aware at the human being through his sensation. The complete makes sense and this sense is not only holistic here spatially but also spectrally. The ontological identity of the structure only takes a meaning dynamically, namely categorically [13]. Therefore, surprisingly, there can be no identity without "temporality"; thus both without scaling and history!

- In the case of a homogeneous partitions this identity signature is written in the space of frequencies by means of a singular TF(n): $Z \sim 1/1+iw\tau$: The Fourier transform of the exponential. The complex field, implying a duality of conjugated variables, is then naturally summoned by the analysis. The spectral signature is based on the inversion of a purely imaginary straight monoidal line (**N**,+) positioned at the abscise "unit: 1" with respect to a reference point of the complex field in stake (Fig. 1). A homogeneous division (here a $1/n$ partition) of the line is nevertheless involved up to infinity. If this inversion is taken with regard to a point A outside the line it brings a compactification, namely a convergence of series bringing infinity within reach. In this case, what is the meaning of the constant "1" of the formulation? The operational calculation clearly tells us that it is the normalized entropy per unit of time and it is associated with the dissipative dynamics represented by the real part R of the exponential relaxation $x \sim exp(-t/\tau) with \tau = 1/RC$ product of a Resistance R and a Capacitance C, i.e. geometrically the parameters of definition of the diameter of the semicircle representing the inversion $1/n$ in the complex plane (distance AO). In addition, the possible adjustment of the base of the logarithm introducing asymptotically *Tropical Geometry*, the universality of the precedence of the spectral analysis over the temporal analysis in all cases of scaling, including non-homogeneous divisions, becomes obvious in spite of its not very intuitive character due to the holistic approach[9].
- The spectral exponential response has the paradoxical advantage of its ambiguity. Indeed, the spectral representation in the complex plane is

[9] This approach can be heard all the more because it can easily show any non-linearities by using the notion of "tone" as in music.

nothing but a geodesic semicircle. From a *cardinal* point of view transformation in (**N,+**):$n \to 1$ matches a "coarse graining" process. In the case of a homogeneous partition, the analysis of the semicircle offers the opportunity for a dual interpretation, one (i) is linear and obvious ($n \to n + 1$) the other (ii) is hyperbolic ($\eta \to \eta + 1$) it is a geodesic functor in a Poincaré models; both are related. A point of the circle is then the formal expression of the perception of the unity vector +1 at the distance n from the origin O, such as an observer could see this unit from the singular point A outer the straight line. The measures are not only reversed but distorted by a parallax (hyperbolic curvature) involved by the cardinal distance n.

- Although the homogeneity assumption hides it, the ambiguity goes far beyond what has just been pointed out. The parametrization constraints rely on an additional duality. There is not only a singular point of view in the problematic but two singular points of view completing each other because by symmetry the two ends of the ½ circle, A and O can be exchanged. In practice there is not one straight line but two straight lines then involved. They can be parameterized in inverse ways if the orientation of the complex plane is fixed. An increase on the right leads to a decrease on the left, a situation which excludes that irreversible time can be associated with the sole representation above. The distance between both possible roots for the referential straight lines is nothing else that the flow of entropy associated to the exponential, namely the irreversibility of time related to the scaling process. Of course, having already studied No ether's works, we expected this observation! In this framework the time is only related to the relaxation process; irreversibility is not intrinsic but related to the scaling associated to the process itself; it is expressed above through the inverse Fourier transform.

We propose to shake up these paradigms by asserting that classes of transfer functions exist which, although *not giving rise to inverse Fourier transforms*, thus to the emergence of a unit of Newtonian time. Nevertheless these classes are physically relevant and equipped with an effective universality for designing a new class of temporality, *another time* [1]. In the framework of Grothendieck's topos, if subtly stacked, these spectra can contribute to introduce a degree of freedom overcoming the absence of inverse Fourier transform then giving birth to new type of proper time (that we call *musical time*). To achieve this program we have altered, falsified and generalized the notion of arithmetic site as described by Connes and Consani [19] by shifting pure and very abstract mathematical concepts into handy and straightforward engineering notions. The idea is to equip the standard spectral exponential with a topos structure by dividing the basic spectrum to give rise to a bi-category mades up of non-finite but complementary hyperbolic geodesics ensuring a physical closure. The set-forcing joins here renormalization techniques. Thereafter a couple of clever unfolding of this topos can give rise to the dual Riemann's zeta functions (with complementary s and $1-s$ exponents) in such a way that these functions can become functorial intercessors both between causal and a-causal "geodesics" and between counting sets and continuous sets. Then our approach makes it possible to design in a categorical framework the question of *the arrow of time* seen as a matching between classes of recursive inverse automorphisms within a bi-scaling closed point of view. This approach requires the understanding of a new concept: the concept of *α-exponential*.

5. Spectral Extension of exponential and α-exponential

Above survey is not a simple reminding of knowledge but the conceptual requisite concerning all fractional dynamics based on universal $Z_\alpha \sim 1/[1+(iw\tau)^\alpha]$, transfer function. We know its dynamical foundation in terms of fractal geometry [17]; for example, $\alpha=1/d$ where d is the non-integer dimension of the immersive geometry. It is used in the definition of the dual closed cantor set with characterized by an α-metric. Traditional apriori concerning PDEs leads quite naturally to think that the universality of this type of dynamics, giving a special place for memory (via convolution) and context (via test functions over the geometry) is questionable[10]. Indeed, the fractional derivative, -and more generally the multiplicity of fractional operators challenging together [18]-, does not simply assign the value zero to any constant, namely to initial conditions. Except for particular metrics this property generates a sensitivity to these conditions. Noether's axioms are then no more relevant leading for example the invalidation of the invariance of energy.

Let us focus first on an emblematic particular case: the derivation of order ½ , the least of fractional operators applications to dispute because it can be linked with spectral exponential without major breaking of symmetries of the complex plane.

[10] Concerning the controversy about fractional operators we recommend the very interesting works of Jocelyn Sabatier and al. (For example *Fractional order derivatives defined by continuous kernel Are they realy too restrictive?* Fractal and fractional 4(3) 40-45, 2020)

5.1. *½-exponential, chance and diffusion*

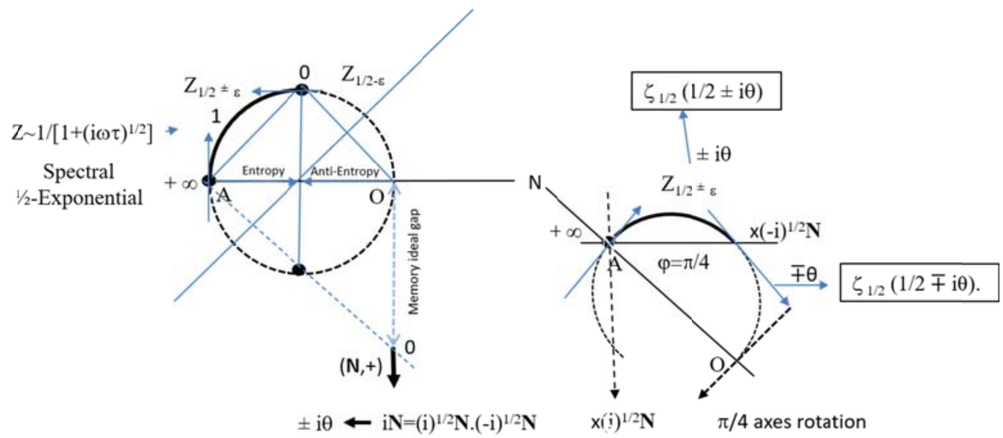

Figure 1. Euclidean (left) (and hyperbolic (Right) representations of Grothendieck dynamic topos based on ½ exponential transfer function, namely transfer of extensive entities across a complete 2D fractal Peano interface [16-17]. The spectrum is obtained via an inversion with respect to A, of the monoid (**N**,+), equipped with a memory-gap at its origin. The figure is characterized by an entropy balanced by ananti-entropy (non-causal component) via a vertical symmetry axis The foliation (fibering) parametrized by θ gives rise to zeta Riemann function. θ plays the role and is analog of the newton time.

It is very easy to skip from the expression of a standard spectral exponential, $Z_\alpha \sim 1/[1+(iw\tau)]$, to the expression of a compactified chance and (finitary) diffusion. It suffices to consider the transfer function $Z_{1/2} \sim 1/[1+(i\omega\tau)^{1/2}]$, namely again, a transformation of a straight line associated to a monoid (**N**,+). However, as shown in figure 1 (right), the line of parametrization must now be tilted at an angle $\pi/4$ with respect to the axes of the complex plane representation. This figure highlitghs, (diagram at right), the reference system specific to fractional dynamics given by above equation (hyperbolic) *versus*, (diagram at left), the same transfer function but represented in a reference system (Euclidean) basing usually the standard exponential. These couple of representations seems mathematically equivalent [11] but they do not hold the same physical meaning. In the "hyperbolic" representation (right) the π/4-tilt of the straight line, that can be symmetrized with respect to spectral frequency limits. This representationis obviously fractionally intrinsic. In "Euclidean" representation (left), though

[11] It is associated to the homographic transformation of the axes of the complex plane

parameterized by a hyperbolic distance, the spectrum, is now represented by the ½-arc of a semi -circle, it is then explicitly referred to the whole semi-circle which corresponds to an inversion of the usual complex line **N**$(0,+\infty)$ along $1+i$. This representation is fractionally extrinsic. The major characteristic of this representation of α–exponential (*if* $\alpha=1/2$) is the following: in external reference (Euclidean left), the ordinate of the origin of the complex plane (the A center of inversion) is partly disconnected from the root of the usual parametrization of the spectral exponential. The representation highlights the presence of *a gap of parametrization*. Conversely in the internal reference frame (Hyperbolic), the gap disappears and it is replaced by a simple tilt with the slope $\pi/4$ at the zero spectral limit (right). Therefore, the non-nil information contained in the gap unveiled in Euclidean representation (left) is compressed via the tilt of the line supporting the parametrization and the punctuation of the spectral fractional representation, is not reduced to a nil value (as "no being") but is a physical emptiness full of hyperbolic non-standard correlations related to the slope of this line. The gap of parametrization can obviously be used as a specific field of inversion with respect to its own root for the monoid (**N,+**) over all the field $(0,+\infty)$. The resulting duality is none other than a specific expression of the quadratic self-similarity affecting the set of integers **N**=**N**x**N**. The meaning of this "zero gap" will appear below in relation with the fractional universality of zeta Riemann function in relation with the mathematical status of the equivalence between both representations.

The item to emphasize at this step of arguing is the following: this specific transfer function has an inverse Fourier transform TF^{-1} which brings into stake both the power law $1/(\pi t)^{1/2}$, as in the processes of infinite diffusion, but also the error function *Erf(t)* by relating the non-finitary scattering process (namely the structure of the gap at root) to closed probability rules with Gaussian distribution of uncertainties. Due to the symmetries involved, the dynamics with parameter $\alpha=1/2$ can then be expressed through a temporal representation using standard Newton time. The emerging time from TF^{-1} is ontologically analogous to that implemented fora standard exponential relaxation. The spectrum from which it results may be amalgamated with an "anti-spectrum" (linked to the "zero-gap" mentioned above) in order to rebuilt a complete exponential giving rise to adual Grothendieck's topos. This spectral topos is endowed with a singularity in its middle. The associated topology is that of a pointed torus whose singular angle at infinity $\varphi$is such as the punctuation is given by $2\varphi=\pi/2$, corresponding ofa permutation of axis of the **C**-plane (the centering of this toposon the complex plane leads to a significant angle at infinity given by $4\varphi=\pi$, confirming le categorical role of the amalgamated semi-circle). We called ½-exponential the spectrum

$Z_{1/2}$~$1/[1+(i\omega\tau)^{1/2}]$, as well as the Newtonian dynamics which only here, can be associated with it[12].

5.2. Generalization at α-exponentials with 1>α>1/2

Although being an obvious generalization of above approachthe spectral analysis expressed by the transfer function Z_α~$1/[1+(iw\tau)^\alpha]$, with1>α>1/2, appears highly complex to interpret physically. At the end of the seventies, for engineering raisons, the TEISI model has had as aim to master this complexity [16]. Just like the simple power law $1/(w\tau)^\alpha$ subject to the convergence constraint $R(\alpha-1)>0$, universal spectra experimentally relevant(transfer function known as Cole and Cole)otherwise characterized by $(\alpha-1)<0$, do not have any inverse Fourier transform[13]. The notion of "standard temporality" (*the idea of an automorphism over the dynamic characterized by only one parameter)* disappears [20]. The representation of the spectrum remains related to the Peano arithmetical rules, for example via the sharing of the discretization of the half-line $m=\omega\tau$, namely via $m \in \mathbf{N}=\mathbf{N}\times\mathbf{N}$. This last constraint preserves the role for the usual symmetrical bi-partitionin the repressentation (namely the role of the phase angle π/4) but this role is now different of the metrical constraints involved by the power law m^α and the dissymmetry: α *versus*1−α. This dual characteristic within the generalization gives to the α-exponential a heuristic indisputable efficiency reinforced by the numerous of experimental data as well in mechanics as in electrodynamics and even in thermodynamics [17]. Unfortunately it is balanced by an increasing level of the complexity for physical understanding. Nevertheless the topology remains that of a torus pointed by an angle $2\varphi=(1-\alpha)\pi/2$. We named α−exponential the dynamics associated with such spectra even if Newton time loses its standard meaning and Noether's principles are no more fulfilled. Without any reference to geometrical scales at this step, the

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[12] It will be observed that the ½-exponential defined above conceals the capacity to transform the operator of succession upon N into a continuous operator. As noted elsewhere by authors **R** and even $\mathbf{R}^n$ do not provide a continuous model of the succession. To be able to do this, it is necessary to use the complex plane, therefore the stacking of **N**using**N**as fiber. A clever stacking of the ½-exponential leading the birth of zeta function appears as the expression of this possibility.

[13] The best proof of the absence of absolute time, and the observation of a requirement for renormalization to bound the dynamic according to the flows of extensities considered. The unit of relevant "time" is a value that becomes a function of this flow in the self-similar context (See J. power Sources 26, 179-185, 1989.)

parameter α expresses nothing other than the inhomogeneous partition mentioned in the introduction. In a context that became "fractal" through recursive partition-iteration, the use of "the Newton time" seen as the ability to express the indexation of the recursive process, -valid in the case in Peano self-similar structure (½-exponential)-, is not authorized. We cannot write any standard differential equation but only formal *fractional differential equations* without any local meaning [16-18]associated with many questions about boundary conditions. Nevertheless "harmonic analysis and transfer function keeps physical meaning in the frame of distribution theory the basis of the TEISI model [16,17]. In particular fractal capacitor behaves like a Choquet non additive capacity. Although the representation of the spectrum is nothing but an arc of a circle, therefore similar to the previous one with $\alpha=1/2$ (figure 2) the asymmetry of the couple of dual geodesical arcs Z_α et $Z_{1-\alpha}$ causes the impossibility of constructing a unique norm over the **N** set valid for the metrical duality, in other words for inversion of the real axis which backs the entropic factor. At least a couple of different norms exists depending on whether the circle is considered either clockwise « α » or anti-clockwise « $1-\alpha$ ». So we discover then surprisingly, that the physical norm to be given to **N** depends on the orientation of the complex plane. We have to switch back the reasoning to understand the strange paradox according to which we are no longer dealing with a pure standard dynamic but with a moving entanglement between experimental non additive measures and the context of these measures. This scientific approach gives a surprisingly engineering content to Schopenhauer idea expressed in: *Die Welt ales Wille und Vorstellung*.

The only relevant relations in this generic bi-partitions featuring a heterogeneous context is given by the α-exponential spectrum and *Scale-Frequency* relations [16]. Both variables are connected through the non-integer dimension $d=1/\alpha$ in accordance with the simplest Mandelbrot 's formula $\eta^d. N\sim1$. Beyond, the writing of **N** as a component in the field of pure complex numbers (TEISI 1978) makes it possible to express without ambiguity the decoupling between the harmonic content of α–process (extensive variable requiring counting and using a unit) and the of Hausdorff weight measure (related to dynamical test function over the fractal geometry therefore an intensive ball of measure requiring another norm). While considering via the product (here expressing an open and unbounded new phase space), the convolution of a physical process with the geometry (monadic unfolding)[14], and its norms must obviously be related via the ontology of the experimental object. For a given α-metric, this complex relation expresses the correlations between a process and a fractal context. If we consider that the magnitude of a given spectral component is a quantity attached to the field **N** (discretely normed with an external clock), the α-spectral analysis takes into account an *n*-indexed set of durations and no longer infinitely precise moments like in differential approach. The discretization becomes intrinsic with respect to the dual analysis and the link to the continuum is, at this step, ignored and entrusted to categorical limit and colimit of the topos only. Accordingly, although without any usual temporality, this approachrequire s the introduction of an external clock for defining the arithmetic unit. Indeed, by taking up the conclusions of Carlo Rovelli [2], according to which physics is not interested in objects but only in their relations, -assertion which implicitly promotes the categorical approach-,any change is thus only relative; any experience even in absence of "temporality » requires the availability of a reference clock, namely an annexed stable *Space-Time*namely: *"any arc of circle requires a circle arc as support"*; an exponential geodesic is required to define a singularity there; a straight line or a plane are required to probe a curvature etc. Annexed pendulum serves only as a metronome (by its escapement mechanism the reference clock is first a metronome and measures a duration before assigning, thanks to its dial, a time to it). The pendulum ensures that any measure of change in any set of systems can be referred to him, thus it becomes a *categorical assistant* in the frame of a reality which must only be handle through a categorical *adjunction* [4-8]. Then, even under the feature of "non-exponential", anyhy perbolic system needs an exponential reference and in addition an external method of compactification to analyzed unbounded process to acquire experimental meaning. The clock no longer defines ordinally the moment *n* but its cardinality *(n,<)* that why any toposrequires the definition of a *classifier* [7,8]. Order classification paves the way for the use of rational set numbers **Q**. At this stage, we can suspect that the continuous is not very far from our issues[15]. Indeed: (i) αhas at least any value between 1 and½. By inversion

[14] Symmetrization is associated with quadratic self-similarity of natural number **N**= **N**x **N**.

[15] The skip from **Q** to **R** corresponds to a completion operation for which there are two approaches either in terms of Cauchy sequences or in terms of cuts (Dedekind) in **Q**. Categorically this double formalism is based on the notions of Cartan filters and ultrafilters. The information conveyed by the zeta function is precisely the existence of these couple of transformations reduced to the scaling properties of **N** seen as a set not above the set of prime

of the monoid (**N**,+) the dynamic is naturally written in **Q**. but one easily understands that any series of symmetrical approximations giving rise to infinite and convergent sequences can lead to more precise data expressible in **R**, but what about diverging series like in power law expressed in the complex plane; (ii) the unity of the *adjunctive approach* even requiring the use of transcendent numbers (π, e, …) naturally asks for the use of the complex field, a result well known by its consequences in quantum mechanics with $\alpha=1/2$; (iii) finally it has been demonstrated that there is an ontological link between the Riemann zeta function and the α-exponential since the latter ensures the unfolding along stalk of the spectral ambiguities created by the set of the infinite number of possible norms and test functions [21-23].

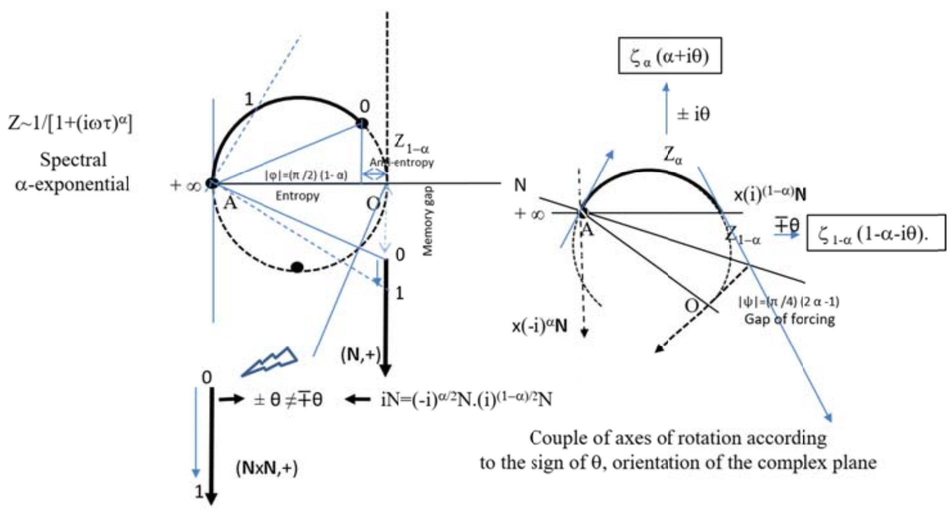

Figure 2. Euclidean (left) and hyperbolic (Right) representations of Grothendieck dynamic topos based on α–exponential transfer function, namely transfer of extensive entities across a non-complete $d=1/\alpha$ fractal interface [16-17]. The spectrum is obtained via an inversion of the monoid (**N**,+), with respect to A if equipped with a memory-gap at its origin. With respect to figure 1 figure 2 is characterized by a breaking of the vertical symmetry. Then the entropy is unbalanced by a non-causal component (anti-entropy). The fibering of each point of the dynamics parametrized by θ gives rise to zeta Riemann function characterizing the dynamic site. The sign ofθ then plays a major role and the time of fibering is different from the Newton time and gives rise to an intrinsic arrow of time. This arrow is related to the difference of unit of monoids with the orientation of the real axis of the complex plane.

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numbers but as declined by the intermediation of filters, of these same numbers (therefore view downstream dual of these)

The authors have shown that the zeta function expresses an interpolation between the discrete set of ordinal counting (entropic) and the continuous set of cardinal measuring (geodesic); this observation is only a natural extension of the theorems of Craig (in logic of predicate) and Pitt and Beck-Chevalley (Category Theory) in the field of experimental constraints. It should be noted that the links between the Riemann zeta function and fractional operators are currently under questions [24,25] but that none of these works are based on category theory even if J. Tenreiro. Machado and Y. Luchko confirmed the link with scaling properties [25] nodal in the TEISI approach extended buy a zeta covering [21-23].

The strangest observation about the dynamic meaning of dual arcs (completion of non-integer geodesics) as dual spectra of fractional dynamics, is the fact that to understand the content of the universal transfer function $Z_\alpha \sim 1/[1+(iw\tau)^\alpha]$, it is necessary to implement the $\alpha-$exponential phase angle $\varphi=(1-\alpha)\pi/2$ that is to say by an easy change of standard axis of the complex plane, to refer the analysis as for stochastic processes (figure 1 right)[16]. The new expression of the phase angle is $\psi=\pm(1-2\alpha)\pi/4$. It becomes symmetrical with respect to anew complex plane orientation and therefore the Grothendieck's topos (requiring finitary limits and cardinal classifier) built on dual "geodesics" using α and $1-\alpha$ becomes implicitly referred to the analytical content of the Riemann zeta function $\zeta(s)$ with $s=\alpha+\imath\omega\tau$, namely its set of non-trivial zeros if Riemann hypothesis is true. The analytical features of this stochastic reference is based on arithmetic quadratic scaling of the product **N=N**x**N**, controlling through $\alpha=1/2$ this set of non-trivial zeros [22, 25-28]. The consequences of the symmetry breaking with regard to stochasticity, namely $\alpha\rightleftharpoons 1-\alpha$ becoming $\alpha+1/2\rightleftharpoons 1/2-\alpha$ is epistemologically radical because with the non-symmetrical structure of the Grothendiek's topos being based on the adjunction $Z_\alpha\rightleftharpoons Z_{1-\alpha}$, the $\alpha-$exponential dynamics matches in the complex plane a projection of the functional zeta equality $\zeta(s) =F[\zeta(1\text{-}s)]$. Due to the θ-covering required for zeta building (complex dimension, scaling and staking are related together), and far from the dynamical morphisms, this functional equation appears as the heart of the *Scale-*

[16] The use of spectral dimensions which gives a referential role to the ½-exponential is due to the fact that this function is the natural receptacle of quadratic self-similarity **N**x**N=N**. We can measure its importance through the symmetrization of the α-exp via the ½ exp. For lack of unfolded singularity on the 1-exp this symmetrization must obligatorily pass through the ½ exp, hence a double game of self-similarity that can only be analyzed through Kan's extension. The stochastic aspect can therefore never be evacuated de l'analyse fractionnaire.

Frequency entanglement. Memory and prophecies could then be the fruits of this entanglement [1].

For giving additional precisions, let us recall that the universal hyperbolic Cole and Cole transfer geodesic namely $Z_\alpha \sim 1/[1+(iw\tau)^\alpha]$, is associated since 1978, with the convolution of a locally linear process (Differential Equation of order one, i.e. a local exponential) and an interface for exchange of extensity equipped with structure a fractal dimension *d* such that $\alpha=(1/d)$ relation is valid only in the simplest case of *Scale-Frequency* coupling but other coupling canal so be considered [29]. Matching with a infinite numbers of possible norms and test functions, the unfolding of dynamic of any irreversible process related to exchanges of extensities (of electrons, ions, momentum, entropy, energy etc.) within a self-similar geometrical context, must be seen as a scan with regard to the scaling. It is expressed via spectra. Initially for experimental reasons, the TEISI, was focused on fractal *Scale-Frequency* coupling law which determines not only an *entropy factor* via the real component of Z_α but also an *anti-entropy factor* via $Z_{1-\alpha}$. This anti-entropic term is associated with internal arithmetic correlations, namely a memory, able to be matched with an virtual inversion of the flow of stochastic time (identical to Newton time) as seen by mean of the phase angle ψ. When the *Scale-Frequency* coupling can be at least partly split, this anti-entropy which is the foundation of any creative processes, appears as the completion to exponential standard[17]. This reference is here due to the local law of exchange that if it can be chosen arbitrarily is often referred to first order processes (Newton time). Therefore, for a given fractal dimension and a given entropic dissipation, the anti-entropy magnitude can always be mastered. This characteristic offers the opportunity for managing the creative process involved by any dissipative process subject to a "wasting time".

6. Quid about "time-gap": toward a wasted time?

The linear interpretation of the irreversibility of time is usually based on an analogy with the coproduct categorical operator defined from the monoidal straight line (**N**,+) but spectral approach of the dynamics the much greater importance of the inversion namely the roles of (**N**,x) and of A. The relation between both monoid appears the issue to be considered, the ordinal scale being the new temporal beat, now equipped with a phase (gap located at the

[17] In fact we come at the next stage with the introduction by two distinct pathways of the zeta function building.

origin of the straight line). This presentation arises the issue concerning the status and characteristics of infinities: what kind of infinity is concerned and thus why kind of zeros? Due to the difference between set and the Cantor set of subsets namely diagonal proof of the existence of different level of infinities, above analysis leads to distinguish countable infinite $\infty_N(Z_\alpha)$ from continuous infinite ∞_c $(Z_{1-\alpha})$. This distinction is valid even for α=1/2 and its symmetrical adjunction based on **N**:**N**=**N**x**N**, namely the content of the octaves. It rises the general question of the structure of the singularity at left and at right. The structure of the dynamical site over the dual and spectral structure of a Grothendieck topos appears as the heart of this issue.

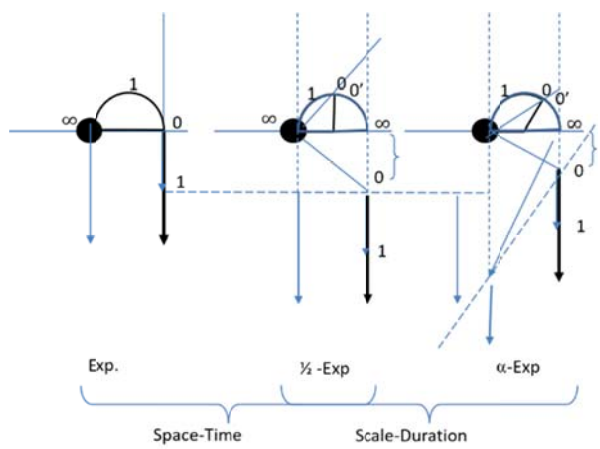

Figure 3. Three possible diagrams of inversion of the semi-infinite straight line with respect to a point external to this semi straight line according to the position of this external point and the value of the norm of the operation of succession in the set of natural numbers **N**.

Why to think of the symmetrization using spectral dimension as the most useful breakthrough for representing the dual Grothendieck's topos $^{1/2}Z_1$ $Z_\alpha \rightleftharpoons Z_{1-\alpha}$? The standard exponential transfer function Z_1 points very precisely, hyperbolic distances u/v~1/(v/u) on the semicircle via the symmetry/inversion of the frequencies [1, ∞] → [1,0]. However, the linearity 1~η.(iωτ) hides the interest of this hyperbolic symmetry that can be also pointed out two time in the topos merging Z_α and $Z_{1-\alpha}$. The insights required for understanding the issue are, up to any is omorphism, based on the positioning of the outer reversal point A and O in the complex plane with regard to the referencing line supporting the both monoid (**N**,+) and the determination of the unit required for the successor operators over both transfer functions. From that three observations must be done:

First observation: the transfer function of the exponential is a whole semicircle, parameterized by $[1,\infty_N]$ therefore directly related to usual arithmetic in **N** *:* $[n \rightarrow n + 1] \rightarrow [n \rightarrow n \pm f_\eta(\pi)]$: i.e. exactly the operation that hides behind the measurement balls required for Fourier or Laplace transformations implementation (Figure 1). Due to the normalization of the lines to be reversed (unit of succession), it is also symmetrized to the standard unity value and u/v ⇌v/u the second part being parametrized over [0,1]. The unit becomes inseparable from the ½ entropy associated to the exponential relaxation (Transfer resistance) expressed mathematically through the Euclidean point/line minimal distance. This means that the flow of entropy is minimal in an irreversible exponential process.

Second observation, bounded stochastic transfer function $Z_{\pm 1/2} \sim \pm 1/[1+(i\omega\tau)^{1/2}]$ associated to diffusion processes, is represented by a couple of half arcs of a circle. The t monoid (**N**,+) axis "temporal straight line" here reversed is characterized by a gap at the origin. The +½-exp spectrum can be completed with a similar arc backed to a vertical symmetry in the complex plane. This symmetrization occurs with an inversion of the orientation of the real axe and a permutation of the imaginary axis and the monoid (**N**,+) axis. The symmetrical group in the complex plane not only has a reference (1, *i*) but also another one obtained by a ± π/4-rotation of the complex plane axes $(\pm i^{1/2}, \mp i^{1/2})$. The phase angle defines the phase locking of the global process expressed by the fractional derivation operator α=1/2. Above observations allows the design of a dynamical "topos" merging causal (positive time) and non-causal (negative time) processes. The symmetry of this topos confirms the relevance of writing the Laplacian operator using a product of ½-operators with $(d^2/dx^2 - Dd/dt) = (d/dx - D^{1/2}\ d^{1/2}\ /dt^{1/2})(d/dx + D^{1/2}\ d^{1/2}\ /dt^{1/2})$ respectively causal (entropic) and non-causal (anti entropic) product. Here anti entropy is only due to uncertainty related to the freedom of referring the octave (**N=NxN**). The limit and co limit are always countable. Entropy and anti-entropy links are kept through all symmetries. Due to the symmetries highlighted above, the sign of the time variable is irrelevant if the physical process is expressed in its topos, namely here with referential of energy[18].

Third observation, the physical meaning of non-causal component may be generalized. The universal transfer function $Z_\alpha \sim 1/[1+(i\omega\tau)^\alpha]$ also occurs itself as an arc of a circle of exponential support. However, in this

[18] If the gap is associated with an inversion of **N** (the field itself associated with time) then the link between the flow of time and entropy (here stochastic) is constrained by the same dynamic norm.

case a part of the previously mentioned symmetries revealed for the½-exp disappear and the symmetrical non-causal completion is now ensured through the complementary spectrum $Z_{1-\alpha}$~-1/[1+$(i\omega_c\tau)^{1-\alpha}$] up to any isomorphism. In this case, the temporal backing of both components ensured by the physical meaning of Laplacian operator, if α=1/2, disappears to reappear rethought and transformed by the categorical approach of the set derivation due to the bipartition $n=\omega\tau$. The role of the Riemann zeta function and the Kan transformation then emerge of the matching of the couple of sets **N** in the bipartition **N**x**N**. Despite obvious differences, above analysis highlights a remarkable categorical filiation between standard exponentiation, bounded stochasticity (½-exp) and fractality (α−exp). This filiation is exclusively determined by the presence or not of a gap at the origins of monoids and by the relative structure of these gaps when ruling the relationship between entropy and anti-entropy. In the frame of simplest TEISI model, characterized by an absence of inverse Fourier transform of transfer function, two methods for introducing the time must then be considered:

A direct way based on the dynamic transfer topos (basis of covering for building the dynamical site). Riemann zeta functionsis constructed via the transformation $1/n^{\alpha} \rightarrow 1/\ n^{\alpha\ +i\theta}$ with the partition $n=\omega\tau$.θ matches or dualizes the time constant τ. REE Communication [22] shows that this operation corresponds to a shift of complex scales with a fractional of order $s=\alpha+i\theta$ on one hand and to a stacking of n via $\theta\in$**N** for a set associated with a quadratic self-similarity **N**=**N** x **N** on the other hand.

An indirect way: as indicated elsewhere, the only contribution of the TEISI model with respect to Benoit Mandelbrot's starting point of view was the introduction (1978) and for engineering reasons [16] of the complex content 1~ $\eta(in)^{\alpha}$. This choice immediately raised the question of the root of the unit namely 1~ $e^{i\theta log(n)}$x $e^{-ki\theta\, log(n)}$, and then open the question of the covering of the topos using ±θas a free external clock with respect to the process. This expression of the root of the unit, leads the introduction of the Riemann zeta function ζ_{α}(s) by using the spectral geodesic extension based on the generic term $1/(n)^{\alpha\pm i\theta}$= $(i)^{\alpha}\eta n^{\pm i\theta)}$ [21]. We can note the "circular" character of the formulation with the complex term "i" appearing both linearly and in the power law therefore when the sign of θ changes, the presence of *a fix* $(i)^{\alpha}$ leads a phase shift which can only be balanced if $(i)^{\alpha}\rightarrow(i)^{1-\alpha}$ namely if ζ_{α}(s) →$\zeta_{1-\alpha}$(1−s)Thus, in the framework of scaling site with $\alpha\neq$ 1/2, the sign of θ plays a role which is no more irrelevant. Then distinct of the possible time constantsτ, it becomes irreversible, involving both the ontological richness of the zeta function and all the difficulties in solving Riemann hypothesis using only analytical approach [23].

The understanding of what temporality means, not only as a flow but also as memory, -here imposed via α-, is then backed to the understanding of the physical meaning of the dual ambiguity concerning the position of the starting axis for the enumeration[19] of cycle, ambiguity also contained in the dual arithmetical expression of the zeta function $\zeta_\alpha(s)$ as additive and product series *in fine* expressed through the functional relation $\zeta_\alpha(s)=F[\zeta_{1-\alpha}(1-s)]$. It is a very old problem of geometry but which we take here in a doubly singular way: first (i) by transforming it into a dynamic problem in d-fractal media with $d=1/\alpha$ (arithmetic approach) whose spectral content is linked to self-similar scaling laws, scaling transposed in a second time, (ii) into the zeta function using a scaling with a complex order $s=\alpha+i\theta$. Let us observe that the θ–unfolding along the stalk plays a role similar to the product $n=\omega\tau$ on the manifold supporting the dynamics. Due to the prime numbers decomposition of n, entangling the variables in a particularly complex way normalization plays here a major role. Let us observe that, θ–stacking then plays a role similar to a magnetic field in the Stern-Gerlach experiment. It helps to unfold the degeneracies hidden in the multi temporal unity of the dynamic. But the stacking along θ-stalk is itself dualized because the straight line expressing the monoid (**N**,+) and its inversion yields in the framework of α–fractional bi-partition, namely with two distinct parts. A part related to the inversion of **N** with respect to the point A outside the line to which must be adjunct the spreading of $m=\omega\tau$, giving a distribution of subset τ by using the fundamental law of arithmetic $m=\Pi_I p_i^{ri}$ with p_i primes number and $r_i \in$ N, namely new form of inversion [$n \rightarrow 1/k$]. This last unfolds in the gap [O,0] located at the root of the straight line to be inversed. The origin is no more a point but an enriched structure that can only be expressed arithmetically and, as Cantor did, unveils a complex set of infinites. The phase angle defines this gap. Due to the existence of a memorization factor that characterizes the boundary, this enlargement at the origin, this rooting is obviously fundamental and deconstructs the "Newtonian time" associated to a squeezing of this gap to a simple singular point. This n-spreading using primes bases the structures of the memory and manifests itself dynamically through the nonzero value of all fractional derivatives [x $(1/i\omega t)^\alpha$] of the

[19] This position requires the extension of the field of **N** to **Q** , the to the **p-adic** field and finally to **R**.

constants [18]. What is important is to understand the dynamic content of this gap which obviously has a direct link both with the quadratic arithmetic self-similarity **N=N x N**, and with the order α of the fractional metrics (locking of the angle phase)[20].

To sum up, the temporal issue in a self-similar bi partition context (α and $1-\alpha$) can be reduced now to give a couple of straight line site parametrized by $n \in$ **N** and compactifies by using an inversion: $[n \rightarrow n + 1] \rightarrow [n \rightarrow n \pm f_\eta(\pi)]$ with respect to any point A located outside of the line such that the norm (namely the 1/2 entropy) be pinned not by means of a single loop (**N**-Norm: $im=i\varpi\tau$) but using a couple of loops (**NxN**-Norm: η_α, and $\eta_{1-\alpha}$). Both dynamics give birth together to a Grothendieck's topos. The needful matching of both norms reveals the origin of the degeneracies of the fractional dynamic, here multi-valuated through the unfolding of the complex planes along a set of stalks giving rise to a sheaf. While Newton time, linearly related to the hyperbolic spectral distance η_α is flowing, many criss-crossings on stalks gives birth to a memory (a multi-valuation due to the discrete self-similarity). The need for steady reference call for an external $i\theta$-parameter ensuring the slicing into an infinite number of stacked washers. Due to the functional relation $\zeta_\alpha(s)=F[\zeta_{1-\alpha}(1-s)]$, the couple of Riemann zeta functions, namely ζ_α(s) and $\zeta_{1-\alpha}$(1–s),don't play the same role. The role of synthetizing above issue is ensured via the zeta function, either analytically via its non-trivial zeros (Riemann hypothesis: $\zeta_{1/2}(s)=0$, $\alpha=1/2$) leading a reversible time that can be assimilate to a Newton time, or categorically via a Grothendieck's topos, taking into account fractional dualization and dissymmetries. Theyradically exclude the reversibility of the propertime (excluding non-trivial zeros $\zeta_\alpha(s)\neq 0$, $\alpha\neq 1/2$) with regard to any type of referential. It will be observed that both loops building the topos, are linked through a dual monadic structure based on the *Scale-Duration* duality, a new fuzzy differential form discretized along scaling.

7. Conclusion about an arrow of time

Paradoxically, thinking about time requires first ignoring it to focus attention on the universal unity of any evolution. Any change must be tamed

[20] Note that the structure of the point at infinity joins the categorical approach concerning the role of ω_1 the first level of non-countable set. The point at infinity of the spectrum is countable in scale but it is uncountable in frequency. It is exactly by reaching this issue that the zeta function acquires its status as an intermediary between the continuous and the countable.

within the framework of category theory using the covering of a Grothendieck topos (initial/sheaves/final) for thinking the related dynamical site. For simplifying due to arithmetic recursion, the topos can be thought as a monadic structure (ultimately as an intermediated causal predicate). Nevertheless, in accordance with Cartesian practices, consciousness enjoins us to consider the the unfolding of this unit. This fragmentation must be ordered according to an indexation in the field of natural numbers *N*, but the ontological unity of the change imposes compactness and therefore the Cauchy convergence of the series leading categorical limits countable and representable. All of these constraints lead to analyze the role and the duality of succession and inversion operators:

- the succession is expressed by means of a normalized half-line [0, potential-infinity] whose additive subsets can be indexed on the fields **N**, **Q** or even but not without cautions via the real set **R**. In **N** the simplest indexing operator is the succession operator $t \rightarrow (t, t + 1)$ but in other field a simple order relation $p < q$ is required. In all fields, a function $f(t)$ can be analyzed from its local derivative $d_t f(t)$ if this last can be defined. This ordinal variable will be termed "the time". It is the most intuitive parameter of the dynamical automorphism for a change controlled via a function.
- Nevertheless, a main duality in discrete indexing in **N** requires the *distinguo* between ordinal (extensive) and cardinal (intensive) character of arithmetic index. This duality gives birth to a complementary of "temporal called "spectral" approach. It consists in to distinguish a counting and a weighting. Since the works of Fourier and Laplace such approach was extended for instance recently in the frame of algebraic geometry, operator algebra and category theory. Roughly it consists in sharing any finite function of $f(t)$, or operator into an indexed sum of cycles (or measuring ball) with increasing normalized radius, namely ν: $\omega\tau = 2\pi\nu\tau$so that the automorphism splits the global unity by means of a series of inclusive sets possibly under-additives. According to a projective categorical approach based on the monoid (**N**,x) the spectral representation operates using an inversion of the ordinal order featured by half-line by using any singular point A outside of it pointing out the duality. Virtually this approach consists in cutting the referencing straight line according to a scaling view [[[[∞]]] infinitely far] so that it implies without additional hypothesis distinguishing two infinities: an infinitely far and countable limit ($t \rightarrow \infty$) but also in addition and beyond an infinitely small, associated with the product operator $\lambda \rightarrow (\lambda, k\lambda)_{k \rightarrow \infty}$ division which

involves the set of subsets an thus open the question of Cantorian infinities $(2^t \rightarrow \infty_c)$. The role of indexation of these subsets with regard to degeneracies and ambiguities resulting of the covering between these sets must be considered using additional tool for example the Riemann zeta function which extend the scanning of indexation using the complex field by bridging countable set and continuous set..

Both approaches are obviously only partially independent and can lead to occurrence an involution. This is the case of the direct and inverse Fourier transforms. However, as seen above, we are much more interested by the absence of involution which ultimately leads to consider a categorical square scheme. Such kind of scheme brings in question K an extensions and diagonal functors. This note asserts that the understanding of temporality can only really be understood from the dynamic site approach as defined above like the dynamical bi-category and it sextension within a square diagram. The "time" as dynamical automorphism parameter only comes second. It emerges if commutativity and involution become relevant.

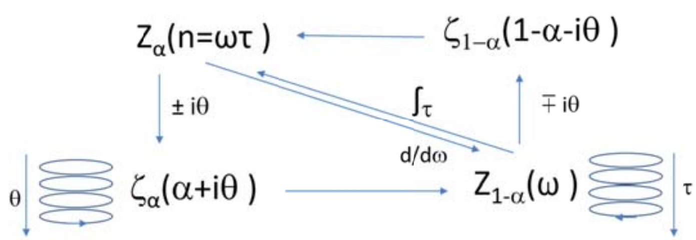

Figure 4. Square diagram characterizing the dynamical site. Causal process is represented at left. The a-causal dual is at right. The foliations below are associated to time like parameters. The upper foliations are spectral like. The diagonal gives rise to a Grothendieck's topos. Th e diagram commutes if and only if α=1/2 (Riemann hypothesis). Due to a phase angle based on α, the proper time of the system θ is reversible if and only if α=1/2.

Beyond this, the morphisms should be analyzed through sheaves weaved above Grothendieck's topos here makes up from the sum of dynamical spectra Z_α and $Z_{1-\alpha}$. For the unique dynamic automorphism parameter, called "time" τ to emerge, there must first exist a norm on the half-line on which the homogeneous ordinal indexing is built. In addition it must exist a cardinal weight $\Lambda \in X$ capable to give birth to an intensive measure of scale $1/\Lambda$. Thus the norm results from an intensity (measurement)/extensity (count) linear relationship. This dual approach gives birth to the *distinguo* usually called "space" and "time". The unit of this dual category can usually be guaranteed by a relation of the type 1~ τ /Λ. The homogeneity that results from the associated splitting up bases the universality of logarithm and exponential applications parametrized by time variable. Nevertheless, as shown above this approach must be capable of generalization for analyzing more complex systems especially chaotic ones. The purely chaotic (Brownian motion, white noise, Markov processes etc.) is 2D stochastics. It responds to probability theories which allows the use of Newton time. This time is given via a clock serving as external arithmetic reference with regard to the process. This last must merge two point of view, the Euclidean one with lengths matches to the clock, and the spectral one matching the scaling adjust on the 2D dial of the clock. In the case of standard stochastic process, the merging is rather simple (Figure 5: ½ exponential).

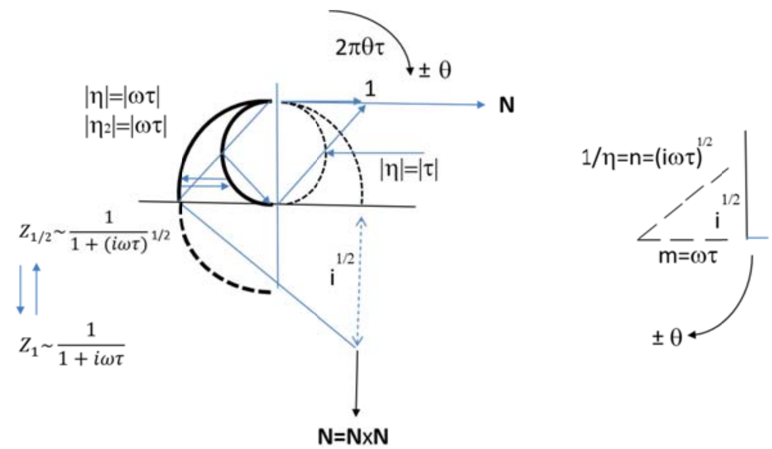

Figure 5. Engineering representation of the dynamic site based on Grothendieck's topos written in the complex plane with θ∈**Z**. The circle of normalization (center) plays a couple of roles: that of a pendulum indexed by θ and partly that of standard exponential spectrum. The external couple of ¼ arc of circle (completion) represents $Z_{1/2}$ in a same dual context. Based on the hyperbolic distance measured on spectra Natural Transforms are constructed between both expansions. $Z_{1/2}$ can be seen as a breach dial of clock whose foliation is clockwise for one and anticlockwise for its completion. These morphisms then extends to the scanning along θ with ±θmod(π). All morphism must respect the rules of arithmetic and especially the fundamental law of sharing on primes set, namely the morphism between breach dial foliated with θand its completion foliated with τ. Due to the power laws usingonly square root both parts are identical and it exists only one spectral mor phism and Euclidean parameter and then an unique order based on θ~τ∈**N=NxN** and the quadratic integer of the phase ±θ~ ∓θ ensures the commutativity of the square diagram of natural transforms. The commutativity is ensures by a simple change of representation axes (rotation ±*π/4*)

It will be observed that the laws of arithmetic make possible to write the self-similar quadratic relation **N=N**x**N** without any change of indexing therefore it exists at least one other class of so-called probabilistic norms such as $1 \sim \tau / \Lambda^2$. This paradigmatic case characteristic of quantum mechanics and statistical thermodynamics relates to $\alpha=1/2$. Referred to the deterministic case using linear space-time relationship, this form expresses a change of octave for the dial size thus a simple change of frame (real axis→ imaginary axis) expressed by a symmetry with regard to $\pi/4$ axis; a simple change of point of view with $\pm (i\tau)^{1/2} \sim \pm\pi\theta$ if $\theta \in$ **N**. Subject to taking the hyperbolic distance, i.e. the inverse of the intensive scale, as a measure of length, θ and $n=\omega\tau$, are commensurable to **N=N**x**N** up to the numerable infinity and thus θ can play the absolute Newton time role (Kan extension is valid). At this stage the model is quite poor closed and in many respects simplistic. Indeed the strength of probabilistic models of uncertainty and white chaos is to fit Noether's principles by adjusting the complex plane axes by implicitly turning the aera of the dial of the clock into a Peano curve. Peano's arithmetic follows. The quadratic nature of arithmetic expressions justifies the adequacy of mechanical action as relevant variable, just as the quadratic nature of energy justifies the invariance of physical laws whatever the sign of the time variable. However, it remains that the process acquires this characteristic only within the framework of a perfectly balanced duality which merges a causal action (affected by an entropy) to an a causal complement. Gaussian thermal fluctuations closes the spectral system upon itself by turning quadratic correlations into numerable and ordered uncertainties, hence the emergence of the Planck factor as categorical limit unit $\hbar=h/2\pi$. Beyond, it will be noted that the uncertainty generated by the fluctuations leads to an uncertainty induced on the metric the underlying dimension being then almost surely itself tainted with uncertainty with $\alpha=1/2\pm\varepsilon$. This metrical uncertainty justifies the Connes-Rovelli hypothesis, their *thermal time arrow* taking its meaning in the analogy between Tomita's theorem applied to von Neumann algebras and the KMS model, asserting herein that temperature T is the unique fluctuations/correlations parameter of thermomechanical issues. The Riemann hypothesis related to the pure metric $\alpha=1/2$ leads to analytical exchange processes, as the set of countable zeros of the zeta function being doubly probabilistic locally via $\hbar$ and globally via $\varepsilon(T)$. Represented by the synthetic diagram below, the case $\alpha=1/2$. is paradigmatic because it expresses both the dynamics of a bounded and complete chaos and the link of this type of chaos with deterministic rules. The time to be used is ultimately standardized over integer metrics in a unique way and the arrow of time can only be "thermal".

Subject to a great deal of precautions, the approach can be generalized by writing $1 \sim m^{\alpha}/\Lambda$ for any geometry with metric $d=1/\alpha$ integer or not (fractal geometry). The extension of figure 5 into Figure 6 highlights precisely the need to distinguish an indexing (extensive-time) from a measurement (intensive-space) even if the quantities must both be written within **N** (TEISI model). While the Newtonian approach implies $\alpha=1$ and the probabilistic approach requires $\alpha=1/2$ the fractal approach for example in the simplest general case corresponds to $1<\alpha\leq 1/2$. The precautions not to be ignored then relate to two items:

(i) Ther equirement for considering the power of the unity which is expressed by means of a dual stacking $1=e^{-i\pi\theta \log(m)}e^{+i\pi\theta \log(m)}$. This relationship involves the dummy variable $\pm\theta$ and the calibration of the dial of the associated clock via $\log(m)$. This calibration adjusts the covering of the dynamic site (topos plus sheaves) onto the design and the building of the zeta function.

(ii) By taking into account the adjunction $\alpha\rightleftharpoons 1-\alpha$ we implicitly introduce completion into the topos by referencing both the Euclidean norm and the probabilistic norm in the framework of a topos by means of pure stochastic dynamics: $\alpha+1/2 \rightleftharpoons -\alpha-1/2$; indeed the above generalization with $m \in \mathbf{N}$ allows to write (TEISI model) the relationship $1\sim m^{\alpha}/\Lambda x\, m^{(1-\alpha)}/\Lambda = m/\Lambda^2$.

It is only at this stage that the importance of the Riemann zeta function really appears as the theoretical intercessor between a given real dynamic (Z_{α}) and the whole field of possibilities attached to it ($Z_{1-\alpha}$) in an arithmetically coherent way. However, we will see that this intersession is not direct. We are going to show that the physical meaning of this real/possible completion leads to the emergence of an arrow of time defined by the behaviour upon the categorical limits. Indeed due to the relative ordered positioning of the dynamic functors Z_{α} *versus* $Z_{1-\alpha}$, as soon as the complex plane, equipped by a dual spectral structure $\alpha\neq 1-\alpha$ an d a critical phase angle becomes a Grothendieck Topos and even before unfolding this topos via sheaves for implementating zeta Riemann function, the sign of θ can no longer be considered as free regarding the choice of the complex plane orientation. Indeed, the unfolding can turn clockwise or anticlockwiseeven if the sign of θ is given. More precisely if $s=\alpha\pm i\theta$, used for the stacking above Z_α gives rise to $\zeta_\alpha(s)$ it follows that que $s=1-\alpha\mp i\theta$. The stacking above of $Z_{1-\alpha}$ gives rise to $\zeta_{1-\alpha}(1-s)$ with inversion of the initial orientation of folding. Hence, the sign is not free. The constraint is due to completion of folding rotation in the spectral adjunction $Z_\alpha \rightleftharpoons Z_{1-\alpha}$ and the clock discrete temporal analysis (different from TF−1) carried out using the operator of rotation $\pm i\theta$, or more precisely the screening operator *exp* $(\pm i\theta\pi log(m))$.

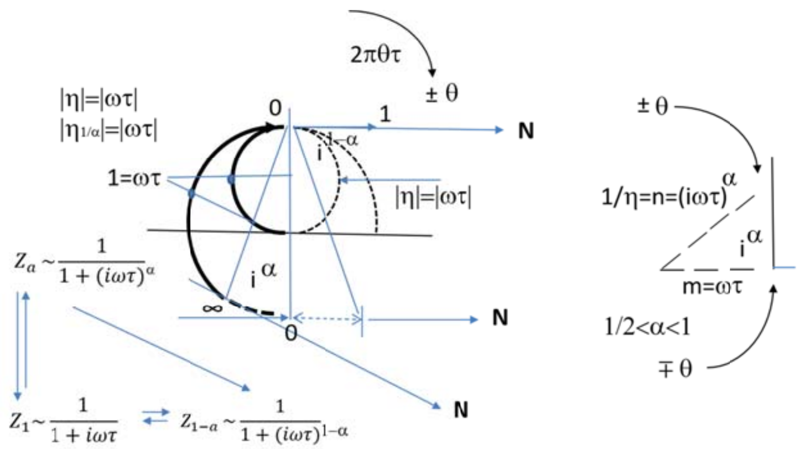

Figure 6. Extension the figure 5 via a symmetry breaking introduced into the Grothendieck's topos for ametric α : ½<α<1. The extrinsic foliation of the dynamic site seen as a stacking of oriented plan complexe parametrized by $\theta\in\mathbf{Z}$ then challenged the intrinsic foliation based on the temporality $2\pi\nu\tau=\omega\tau$ attached to $Z_{1-\alpha}$ via a parametrization using a proper time τ. The commutativity of natural transform the square diagram is then broken, as long as the arithmetic rules are fulfilled.

Moreover the related morphisms require, among other things, to put in phase infinities and zeros, operation which requires to adjust specifically the current infinity carried by the phase angle and the space-time units. It is at this stage that the adjunction between the spectralα-dynamics according to the order of the scales η, Z_α and the set of time constants τ becomes interesting accordingly with Kan extension. Indeed, by taking into account the bi-partition of $m=\omega\tau$ associable to the dynamic functor $Z_{1-\alpha}$, the set of τ is associated with a set ofgreat number of fractal structures in **N** under the contraint $d=1/\alpha$.

If $\pm\theta$ account for the Newton's time, that of an external clock, the geometric structure of adjunction $\alpha \rightleftharpoons 1-\alpha$ (Internal/external; real/possible) imposes the existence of a field of correlations which goes be explored using $\pm\theta$. In the discrete logic of the TEISI model which distinguishes the hyperbolic distance as a measure $1/\eta \sim (\omega\tau)^{\alpha}$ and the indexing of this measure by a quantity analogous to a time τ it is important to count and qualify the set of the bipartitions $m=\omega\tau$ likely in **N=NxN** to give rise to an index τ. It is via the completion hidden in the adjunction $\alpha \rightleftharpoons 1-\alpha$ therefore via the functional relationship characterizing the zeta function that the arrow of time is then surreptitiously introduced. Indeed, knowing that then $dm^{\alpha}/d\omega=\alpha\tau^{\alpha}(1/\omega)^{1-\alpha}$, the fundamental equation of the model gives, $\forall\theta$: $i^{\alpha}.\eta.e^{\mp i\theta log(m)} \sim m^{-s}$ with $s=\alpha\pm i\theta$, i.e. the generic term of the zeta function $\pm i\theta$, has the role of leafing through in a trivial way over the only field of octaves the real dynamics (Z_{α}). To do this, we use the order relation concerning η^{d}, and in a subtle way by decomposition on ideal structures, those of possibilities $Z_{1-\alpha}$. In order to ensure a zeta coherence, the foliation of the real spectral dynamics is ensured in the field of integers $m \in$ **N** normed by the factor $log(m)$ which fixes the extension of the spectrum reduced to the modulus of rotation in the complex plane. The factor θ systematically tests the set $\{m\}$ considered as integers while keeping the phase angle constant $\varphi=(1-\alpha)\pi/2$. This foliation matches the covering of disks of increasing size without the interest of the zeta function appearing clearly at this stage. However, every disk can itself be leafed in response to the possible bi-partitions of m. This second level of foliation must be represented by means of ($Z_{1-\alpha}$) and ordered by means of $\{\tau\}$ associated with a new hyperbolic distance $\eta=\square(1/\omega)^{1-\alpha}$. This is tested via the discontinuous map: $m \rightarrow \{1/\omega\}^{1-\alpha}$. Given this constraint, the power law $\mp\theta$ is then used as the suitable clock for the dynamics (internal clock).The question is then to find the number of cycles $\{\tau\}_m$ necessary to phase all the possible m bipartitions The catenation of the sheaf parameterized by $m \in$ **N**, leads to the replacement of the stack $\{m\}$ by a sheaf of tangled "spaghetti" whose sections give the levels of correlations for a certain set of m. The $\aleph_0$-countable infinity of cycles parametrized by $\pm i\theta$ will give rise to an uncountable $2^{\aleph_0}$ infinity of cycles parametrized by $\mp\theta$, so that the ontological unity of the final/sheaf/ initial site is guaranteed. This is exactly what Voronin's theorem and Bagchi's lemma also teach us about to approximate the distribution of zeros of the zeta function. The non-trivial zeros of the zeta function arise precisely here as an ability to deal analytically with probabilistic chaos in accordance with a symmetrization of the dynamic automorphism parameter. This property suggests treating the general case in such a way as to make $\alpha=1/2$ appear as the reference metric. Thus we assert here that the reference time is always the stochastic time $\pm\theta$ associated with $\alpha=1/2$ capable to take into account the phase angle $\pi/4$. It is then from this axis of the complex plane that the introduction of the irreversibility of the clock time $\tau=\pm i\theta_{1/2}\mp\Delta_\theta$. is explained. More explicitly in the case $\alpha\neq 1/2$, we can write $\alpha=1/2+\beta$ et $1-\alpha=1/2-\beta$ the function $\zeta_\alpha(s)$ introduces the factor $n^{\pm i\pi\theta}$, namely $exp[\pm(2/(1+2\beta))i\pi\theta\, log(m^{\alpha})]$ while $\zeta_{1-\alpha}(1\text{-}s)$ introduces the factor$\omega^{\mp i\pi\theta}$, that is $exp[\pm(2/(1-2\beta))i\pi\theta\, log(\omega^{1-\alpha})]$. Thus, due to the fact that the dynamics cannot be referred to an axis of symmetry for $\alpha\neq(1-\alpha)$ any change of orientation of the complex plane with respect to an axis of symmetry $\pm i\theta\rightarrow\mp\theta$ must take into account has a phase shift angle $|2\psi|=|2\alpha-1|\pi/2$. Admittedly the beat of an external clock $exp(2i\pi\theta)$ therefore $\pm i\theta$ always makes it possible to qualify and count the commensurable space/time correlations but this beat is affected by a phase which subsists at the limits of the basic Topos $\psi(\pm\theta)=-\psi(\mp\theta)$. The inversion of the sign θ, of scanning without consequences in terms of commensurability in the adequate metric $\alpha=1/2$, acquires in fractal metric $\alpha\neq 1/2$ a decisive impact by causing a symmetry breaking. Not only do the direct and inverse rotations not concern the same categorical object (real dynamics Z_α / possible dynamic $Z_{1-\alpha}$) but also do not concern the same class of set ($\aleph_0$–countable versus $2^{\aleph_0}$–continuous). This "dual duality" is hidden, in the case $\alpha=1/2$ which led one of the authors (P. Riot) to equate Riemann's hypothesis to Martin's axiom in set theory namely $\boldsymbol{\omega_1}: 2^{\aleph_0}\ \tilde{}\ \aleph_0$[30]. This axiom amounts to assuming that the gap between the countable and the continuous can be reduced to a Peano surface in other words a bi-partition based on **N=NxN**. Any complete symmetric bipartition is thus associated with the Cantorian metric $\alpha=1/2$, each of the components then being fiberized in a numerable way, giving rise to an analytical approximation (Voronin's theorem and Bagchilemma [22]).

The general case $\alpha\neq 1-\alpha$ based on the monadic structure schematized in figure 7 could lead to think on the one hand that the ordered gap at infinity (between countable and continuous) could be bi-partitioned asymmetrically with a mirror $\boldsymbol{\omega_1}$ and on the other hand that a permutation of the direction of rotation due to a functorial reflection could be take place in the gap with respect to $\boldsymbol{\omega_1}$ mirror. This reflection is not thinkable because it would suppose an inversion of the relation of order of complexity between the countable and the continuous, able to remove the arbitrary character of the sign of θ while preserving at the limits the categorical distinction between *countable limit* $\alpha\rightleftharpoons 1-\alpha$ *co-limit continuous* but by forcing the system to adopt a statistical chaos. Thus a quantum of entropy $\boldsymbol{\varepsilon_S}$ appears at infinity which practically manifests itself by the appearance of an arrow of time that propagates the monad at all scales of functorization through the *distinguo* θ (reference clock time) and τ (the parametrization of the proper time). The difference is the universal *engine of the time*. Irreversibility is exhibited by the absence of commutativity of the square diagram linking the categories of dynamic Topos and the Riemann functions associated to that of the site based on it (Figure 4). This absence is due more precisely to the characteristics of the diagonal of the square diagram which, like in the case of Cantor's demonstration highlights the existence of several classes of infinities, points the absence of equivalence between the test of the distribution of τ by means of θ, and the reverse test of the using-θ by means of the same τ. As it had already been pointed out the functional commutator is then different of zero: $[\hom(Z_\alpha, Z_{1-\alpha})-\hom(Z_{1-\alpha}, Z_\alpha)]\neq 0$. This absence is precisely due to the difference in correlations reducible to transpositions to octaves, i.e. **N=NxN**, and correlations linked to the fractional metric associated with a tensorial monad over scaling. The commutativity is restored by Martin's axiom when $\boldsymbol{\omega_1}$ is a mirror between countable and continuous namely for $\alpha=1/2$ (homogeneous bi partitions), then justifies [30] the reduction of the Riemann hypothesis to Martin's axiom and the transfer between quantum level to a countable statistic over the field of prime numbers.

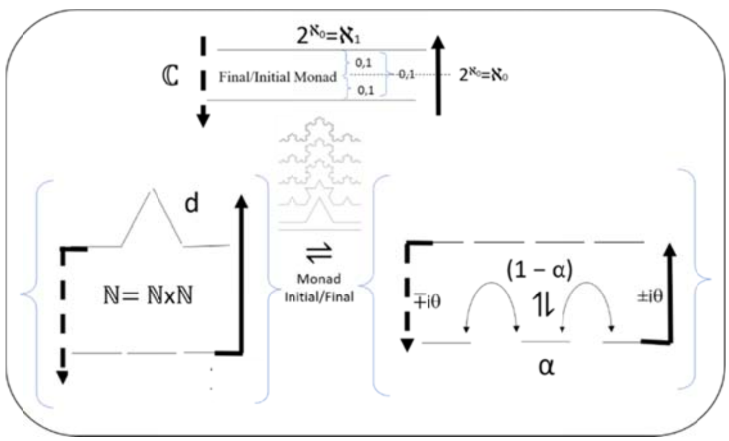

Figure 7. Synthesis of the geometrical origin of dual foliation created by fractal dynamic (TEISI model) expressed using the Grothendieck topos based on Z_α and unveiled via $\zeta_\alpha(s)$ Riemann function. This schematization highlights and expresses the open issue at infinity and significance of Martin's axiom for matching the Riemann hypothesis.

To sum up, as Rovelli points out, any change only makes sense relative to an external reference. It follows that time must be dual. This duality is revealed through dynamic sites capable of distinguishing between a reference clock beat and its dual basis as Grothendieck's topos. The double duality that emerges from the multiple bi-partitioning capacities of the real is veiled in deterministic systems but it emerges in complex systems through the tensor structure between the proper time/clock time. Analyzed in terms of clock time, the proper time reveals a contra-variant component corresponding to the irreversibility of time as observed experimentally, in particular in recursive dynamics and self-similar media. This irreversibility is not only linked to the phenomenon serving as a test but partly linked to the topology and more precisely to the metric of dual-time structure analogous to a space-time. Constraint at categorical limits, this topology leads to the existence of an arrow of time which must be associated with a necessary entropic adjustment of the complexity over a dual infinity (countable and continuous). Indeed, at this stage the clock time and the proper time can only be constrained by the self-similarity of the countable **N=NxN** which arithmetically assimilates probabilistic time and deterministic time as for example in Quantum Mechanics. Mathematically, the set of zeros of zeta Riemann function and the Martin's axiom make this constraint explicit, bringing thus the arrow of time back to a still open hypothesis.

In Memoriam: This note was written in memory of my friend José Tenreiro Machado, whose dynamism and scientific relevance I have always admired [31-35]. Rest in peace!

Author Contribution: PR is currently studying the pure categorical content of zeta function design [30]. ALM develop the idea of time starting unfolding of Grothendieck's topos through zeta function building and has written the draft of this note. AG has help him to relate fractal geometry to atomic topology and its link with quantum mechanics [11]. These studies was developed in the frame of Complex System Group implemented by DT in the Physic department of KFU [36].

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