COMPLEXITY

ON DISCRETE FUNCTIONS AND REPETITIVE ARRANGEMENTS. ALGORITHMS TO CONSTRUCT ALL DISCRETE FUNCTIONS. PRACTICAL INTERPRETATION

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Abstract. *The main idea of this work is based on the question: how can we control the electric circuits between a number of electric bulbs and a number of electric sources. This idea generates the correspondences between two discrete sets. The correspondence is based on the notion of discrete function and repetitive arrangements.*

The normal construction and notions of the work are introduced gradually and are detailed at every stage. Our constant endeavor has been to ensure that every sentence in the work has a logical position.

Here appears many questions: how to construct all discrete functions, which is the total number of these functions, which is the relation between the number of bulbs and the number of sources, can we construct and control only a partial number of electric circuits (by direct access method) etc.

The work answers all these questions by specially algorithms: the construction algorithm and the decomposition algorithm. The algorithms use the rule from left to right to construct all possible discrete functions and, hence, all electric circuits. The decomposition algorithm supplies an access direct method. So we can control any part of the whole set of circuits.

A lot of notions and specific notations are used to develop and illustrate the work. For combinations we have to show the construction elements. A lot of examples explain this important notion.

The work contains a lot of numerical examples and applications.

The last section of the work deals with the bijective (and invertible) functions. Specialized notions and notations are used. Numerical examples and geometric designs illustrate the theory.

Keywords. *Arrangements, repetitive arrangements, discrete functions, decomposition algorithm, rule left right.*

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1. A practical problem

How to control k electric light bulbs $\{ a_1, a_2, \cdots, a_k \}$ alimented (supplied) by n $\{ b_1, b_2, \cdots, b_n \}$ sources of electricity. To control it means to know the form of electric circuit, namely the correspondence between these two sets.

2. Introduction. Notations

In order to solve the above problem we use the discrete functions [1], [3], [8].

We denote $f : A \to B$, A-domain, B-codomain; with the discrete sets

$$A = \{ a_1, a_2, \cdots, a_k \}, \quad B = \{ b_1, b_2, \cdots, b_n \}$$

for any natural non-null numbers k and n; $card\, A = |A| = k$, $card\, B = |B| = n$, $a_i \neq a_j$ for any i and j. The set B is a multiple set [11].

The set of all functions $f : A \to B$ has the cardinal

$$card\{ f \} = card\, B^{card\, A} = n^k \qquad (*)$$

In applications we use $A = \{ 1, 2, \cdots, i, \cdots, k \}$, $B = \{ 1, 2, \cdots, j, \cdots, n \}$.

There are three cases: $k < n$, $k = n$, $k > n$.

The injective functions f could be obtained for $k \leq n$.

The surjective functions f could be obtained for $k \geq n$.

The bijective functions f could be obtained for $k = n$.

Our aim is to construct all n^k functions $f : A \to B$. Then we will analyse the bijective functions.

Remark 1. Permutations and arrangements are ordinate sets, while combinations are subsets of a set.

We present a short comparison between the usual arrangements of n objects taken k at a time and the repetitive arrangements (arrangements with repetition) of n objects taken k at a time.

Usual arrangements. Example 1. $B = \{ 1, 2, 3 \}$; $n = 3$, $k = 2$; $n \geq k$ (always).

Permutations $n = 3$; 123; 132; 213; 231; 312; 321; $n! = 3! = 6$.

Arrangements of $n=3$ taken $k=2$; 12; 21; 13; 31; 23; 32; $A_n^k = A_3^2 = 6$.

Combinations $n=3$ taken $k=2$; 12; 13; 23; $C_n^k = C_3^2 = 3$.

Repetitive arrangements [2]. Example 2. $B=\{1,2,3\}$; $n=3$, $k=2$ (any value n,k).

11; 12; 13; 21 22; 23; 31; 32; 33; $n^k = 3^2 = 9$. Denote $n^k = N$.

Example 3. $B=\{1,2\}$; $n=2$, $k=3$; $n \le k$.

111; 112; 121; 211; 122; 221; 212; 222; $n^k = 2^3 = 8$.

Another notation for the total number of all repetitive arrangements is $\overline{A}_n^k = n^k = a_n^k = N$.

3. Problem formulation

Related with this work we have two aims.

I) Aim 1. We have to construct **the set of all discrete functions** $f: A \to B$, $f: \{a_1, a_2, \cdots, a_k\} \to \{b_1, b_2, \cdots, b_n\}$ for any non-zero natural numbers n,k.

The total number of these functions is denoted $\overline{A}_n^k = n^k = N$.

There are several methods to construct all discrete functions f.

We propose a method based on direction **left to right** in the set B, and elaborate **the algorithm left – right** (algorithm 1). It is a sequential method.

II) Aim 2. We **make a decomposition** of $\overline{A}_n^k = n^k$ based on **the decomposition algorithm** (algorithm 2). It is a direct access method, i.e. we can construct any subset of the hole set with $\overline{A}_n^k = n^k$ functions.

Examples. Some particular cases and the total number of functions.

Nr	k	n	n^k	$f: A \to B$	
1	1	2	nu	1 $\to$ 1,2	$N=2$.
2	2	1	1	1, 2 $\to$ 1	$N=1$.
3	2	2	4	1, 2 $\to$ 1,2	$N=4$.
4	3	2	8	1, 2, 3 $\to$ 1,2	$N=8$.
5	4	2	16	1, 2, 3, 4 $\to$ 1,2	$N=16$.
6	5	2	32	1, 2, 3, 4, 5 $\to$ 1,2	$N=32$.

7	2	3	9	1, 2 $\to$ 1, 2, 3	$N = 9$.
8	3	3	27	1, 2, 3 $\to$ 1, 2, 3	$N = 27$.
9	4	3	81	1, 2, 3, 4 $\to$ 1, 2, 3	$N = 81$.
10	5	3	243	1, 2, 3, 4, 5 $\to$ 1, 2, 3	$N = 243$.
11	2	4	16	1, 2 $\to$ 1, 2, 3, 4	$N = 16$.
12	3	4	64	1, 2, 3 $\to$ 1, 2, 3, 4	$N = 48$.
13	4	4	256	1, 2, 3, 4 $\to$ 1, 2, 3, 4	$N = 256$.
14	5	4	1024	1, 2,3, 4, 5 $\to$ 1, 2, 3, 4 etc.	$N = 1024$.

Remark 2. The total number of functions increases very quickly with *k* and *n*.

4. The Algorithm left-right to construct all discrete functions

The functions are $f : A \to B$, $A = \{a_1, a_2, \cdots, a_k\}$, $B = \{b_1, b_2, \cdots, b_n\}$, where $k \geq 1$ and $n \geq 1$ are given natural numbers.

The total number of functions f is the total number of repetitive arrangements.

The rule from left to right [8] means: if the fix value is *j* or b_j, then, for combinations one uses only the values $S = \{b_{j+1},, b_{j+2},, \cdots, b_n\}$ from the right part of b_j.

Remark 3. The algorithm is based on combinations and the moving rule down-up.

Example. 1) 1,2,3 $\to$ 1,2,3; C_3^1 =3;

$f(1) =$ 2 1 3;
$f(2) =$ 1 3 2;
$f(3) =$ 3 2 1.

2) 1,2,3 $\to$ a,b,c,d; C_3^1 =3;

$f(1) =$ b b d;
$f(2) =$ b d b
$f(3) =$ d b b.

4.1. Examples of all discrete functions for small numbers k and n

Application 4.1. We use the algorithm 1. It is a sequential method. (**Version 1**).

1) $f: 1, 2 \to 1,2$, $N = 2^2 = 4$

Nr	1	2	3	4
$f(1) =$	1	1	2	2
$f(2) =$	1	2	1	2

2) $f: 1, 2,3 \to 1,2$, $N = 2^3 = 8$

Nr	1	2	3	4	5	6	7	8
$f(1) =$	1	1	1	2	1	2	2	2
$f(2) =$	1	1	2	1	2	2	1	2
$f(3) =$	1	2	1	1	2	1	2	2

On the line Nr we count the current number of function f or its address.

3) $f: 1, 2,3,4 \to 1,2$, $N = 2^4 = 16$ functions (small number).

Nr	1	2	3	4	5	6	7	8	9	0	1	2	3	4	5	16
$f(1) =$	1	1	1	1	2	1	1	2	2	1	2	1	2	2	2	2
$f(2) =$	1	1	1	2	1	1	2	2	1	2	1	2	2	2	1	2
$f(3) =$	1	1	2	1	1	2	2	1	1	1	2	2	2	1	2	2
$f(4) =$	1	2	1	1	1	2	1	1	2	2	1	2	1	2	2	2

4) $f: 1, 2 \to 1,2,3,4$, $N = 4^2 = 16$ functions (small number).

Nr	1	2	3	4	5	6	7	8	9	0	1	2	3	4	5	16
$f(1) =$	1	1	2	1	3	1	4	2	2	3	2	4	3	3	4	4
$f(2) =$	1	2	1	3	1	4	1	2	3	2	4	2	3	4	3	4

5) $f: 1, 2,3,4,5 \to 1,2$, $N = 2^5 = 32$ functions (small number).

Nr	1	2	3	4	5	6	7	8	9	0	1	2	3	4	5	16	17	8	9	0	1	2	3	4	5	26
$f(1) =$	1	1	1	1	1	2	1	1	1	2	2	1	1	2	1	2	1	1	2	2	2	1	2	1	2	2
$f(2) =$	1	1	1	1	2	1	1	1	2	2	1	1	2	1	2	1	1	2	2	2	1	2	1	2	2	1
$f(3) =$	1	1	1	2	1	1	1	2	2	1	1	2	1	2	1	1	2	2	2	1	1	1	2	2	1	2
$f(4) =$	1	1	2	1	1	1	2	2	1	1	1	1	2	1	1	2	2	2	1	1	2	2	2	1	2	1
$f(5) =$	1	2	1	1	1	1	2	1	1	1	2	2	1	1	2	1	2	1	1	2	2	2	1	2	1	2

Nr	27	8	9	0	1	32
$f(1) =$	1	2	2	2	2	2
$f(2) =$	2	2	2	2	1	2

$f(3) =$ 2 2 2 1 2 2
$f(4) =$ 2 2 1 2 2 2
$f(5) =$ 2 1 2 2 2 2

6) f : 1, 2,3 $\rightarrow$ 1,2,3, $N = 3^3$ =27 functions (small number).

Nr	1	2 3 4	5 6 7	8 9 0	1 2 3	4 5 6	7 8 9	0	1 2 3	4 5 6 7=27
$f(1) =$	1	1 1 2	1 1 3	1 2 2	1 2 3	1 3 2	1 3 3	2	2 2 3	2 3 3 3
$f(2) =$	1	1 2 1	1 3 1	2 2 1	2 3 1	3 2 1	3 3 1	2	2 3 2	3 3 2 3
$f(3) =$	1	2 1 1	3 1 1	2 1 2	3 1 2	2 1 3	3 1 3	2	3 2 2	3 2 3 3.

Remark 4. For $k = 3$ and $n = 3$ a short summary of the basic idea of computations has the following form

$f(1) =$	1	; 1	; 1	; 2	; 2	; 2	; 3
$f(2) =$	1	; 1	; x	; 2	; 2	; x	; 3
$f(3) =$	1	; x	; y	; 2	; x	; y	; 3

$C_3^3(1;\theta)$; $C_3^2(1;x)$; $C_3^1(1;x,y)$; $C_3^3(2;\theta)$; $C_3^2(2;x)$; $C_3^1(2;x,y)$; $C_3^3(3;\theta)$.

The symbol θ is empty set and $x, y, \cdots \in S$.

4.2. Examples of all discrete functions for great numbers k and n

We use a modified sequential method (**Version 2**).

Use successively the rules a), b), c), d), e).

a) Use the subset $S \subset B$ with all elements from the right part of j.

b) Denote by L the completion length, $0 \le L \le k-1$.

c) Construct all repetitive arrangements of S having the length L, $L = 0,1,2\cdots$ etc.

d) Compute the total number of arrangements $(n-j)^L$. Denote $(n-j)^L = R.$

e) Increase the last address with the value $R\,x\,C_k^{k-L}$.

Construct all bijective functions corresponding to the number $(n-j)^L$.

Application 4.2.1. f : 1,2,3 $\rightarrow$ 1,2,3, 4, 5, $k = 3$; $n = 5$; $N = 5^3$ =125 functions (great number).

Step $j = 1,\ L = 0;\ C_k^{k-L} = C_3^{3-L} = 1.$

No 1 (first address)

$f(1) = 1$

$f(2) = 1$

$f(3) = 1$

$j = 1,\ L = 1,\ S = \{\ 2,3,4,5\ \},\ (n-j)^L = (5-1)^1 = 4 = R;$ use 2 3 4 5 and the arrangements 2, 3, 4, 5; $C_k^{k-L} = C_3^{3-L} = 3.$

Increase the last address 1 with the value R x3=12; 1+12=13; true.

No 2 3 4 5 6 7 8 9 10 11 12 13 (addresses)

$f(1)$ =1 1 2 1 1 1 1 5

$f(2)$ =1 2 1 1 1 1 5 1

$f(3)$ =**2** 1 2 **345** 1 1

$j = 1,\ L = 2,\ S = \{2,3,4,5\},\ (n-j)^L = (5-1)^2 = 16 = R;$ use 2 3 4 5 and the arrangements

2 2 2 2 3 3 3 3 4 4 4 4 5 5 5 5; $C_k^{k-L} = C_3^{3-L} = 3.$

2 3 4 5 2 3 4 5 2 3 4 5 2 3 4 5;

Increase the last address 13 with the value R x3=48; 13+48=61; true.

No 14 15 16 17 18 19 20 21 22 23 24 25 ⋯⋯ 56 57 58 59 60 61 (addresses)

$f(1)$ =1 2 2 1 1 1 3 2 1 5 4 1 5 5

$f(2)$ =**22** 1**223** 2 1 **5** 4 1 **5** 5 1

$f(3)$ =**21** 2**342** 1 3 **4** 1 5 **5** 1 5

Step $j = 2,\ L = 0.\ C_k^{k-L} = C_3^{3-L} = 1.$

No 62 (new address)

$f(1) = 2$

$f(2) = 2$

$f(3) = 2$

$j = 2,\ L = 1,\ S = \{3,4,5\ \},\ (n-j)^L = (5-2)^1 = 3 = R;$ use 3,4,5 and the arrangements 3, 4, 5; $C_k^{k-L} = C_3^{3-L} = 3.$

Increase the last address 62 with the value R x3=9; 62+9=71; true.

No 63 64 65 66 67 68 69 70 71 (addresses)

$f(1)$ = 2 2 3 2 2 2 5

$f(2)$ = 2 3 2 2   2 5 2
$f(3)$ = **3** 2 2 **45** 2 2

j =2, $L=2$, S ={ 3,4,5 }, $(n-j)^L$ $=(5-2)^2=9$ $=R$, use 3, 4, 5 and the arrangements

3 3 3 4 4 4 5 5 5; C_k^{k-L} = C_3^{3-L} = 3.
3 4 5 3 4 5 3 4 5;

Increase the last address 71 with the value R x3=9; 71+27=98; true.

No 72 73 74 75 76 77 78 79 80 ⋯⋯.93 94 95 96 97 98 (addresses)
$f(1)$ =2 3 3 2   2 3 5 ⋯⋯ 2   2 5 5
$f(2)$ =**3** 3 2 **33** 5 2 ⋯⋯   **55** 5 2
$f(3)$ =**3** 2 3 **45** 2 3 ⋯⋯   **45** 2 5

Step j =3, $L=0$; C_k^{k-L} = C_3^{3-L} = 1.

No 99 (new address)
$f(1)=3$
$f(2)=3$
$f(3)=3$

j =3, $L=1$, S ={4,5}, $(n-j)^L$ $=(5-3)^1=2$ $=R$; use 4,5 and the arrangements 4, 5; C_k^{k-L} = C_3^{3-L} = 3.

Increase the last address 99 with the value R x3=6; 99+6=105; true.

No 100 101 102 103 104 105 (addresses)
$f(1)$ =3 3 4 3 3 5
$f(2)$ =**3** 4 3 **3** 5 3
$f(3)$ =**4** 3 3 **5** **3** 3

j =3, $L=2$, S ={4,5}, $(n-j)^L$ $=(5-3)^2=4$ $=R$; use 4,5 and the arrangements 4 4 5 5 ; C_k^{k-L} = C_3^{3-L} = 3.

4 5 4 5;

Increase the last address 105 with the value R x3=12; 105+12=117; true.

No 106 107 108 109 110 111 112 113 114 115 116 117 (addresses)

$f(1)$ = 3 4 4 3 4 5 3 3 5 5
$f(2)$ =**4** 4 3 **4** 5 3 **55** 5 3
$f(3)$ =**4** 3 4 **53** 4 **45** 3 5

Step j =4, $L=0$; C_k^{k-L} = C_3^{3-L} = 1.
No 118 (new address)
$f(1)=4$
$f(2)=4$
$f(3)=4$
j =4, $L=1$, S ={5}, $(n-j)^L$ $=(5-4)^1=1$ $=R$; use 5
and the arrangements 5; C_k^{k-L} = C_3^{3-L} = 3.

Increase the last address 118 with the value R x3=3; 118+3=121; true.

No 119 120 121 (addresses)
$f(1)$ =4 4 5
$f(2)$ =4 5 4
$f(3)$ =**5** 4 4
j =4, $L=2$, S ={ 5 }, $(n-j)^L$ $=(5-4)^2=1$ $=R$; use 5
and the arrangements 5; C_k^{k-L} = C_3^{3-L} = 3.
5 ;
Increase the last address 121 with the value R x3=3; 121+3=124; true.

No 122 123 124 (addresses)
$f(1)$ =4 5 5
$f(2)$ =**5** 5 4
$f(3)$ =**5** 4 5

Step j =, $L=0$; C_k^{k-L} = C_3^{3-L} = 1.
No 125 (last address). End of computing. Stop. N =125.
$f(1)=5$
$f(2)=5$
$f(3)=5$

Application 4.2.2. $f:$ 1, 2,3,4,5 $\to$ 1,2 , N =32

No 1 2 3 4 5 6 7 8 9 0 1 2 3 4 5 16 17 8 9 0 1 2 3 4 5 6=26

$f(1) =$ 1 1 1 1 1 2 1 1 1 2 2 1 1 2 1 2 1 1 2 2 2 1 2 1 2 2

$f(2) =$ 1 1 1 1 2 1 1 1 2 2 1 1 2 1 2 1 1 2 2 2 1 2 1 2 2 1

$f(3) =$ 1 1 1 2 1 1 1 2 2 1 1 2 1 2 1 1 2 2 2 1 1 1 2 2 1 2

$f(4) =$ 1 1 2 1 1 1 2 2 1 1 1 1 2 1 1 2 2 2 1 1 2 2 2 1 2 1

$f(5) =$ 1 2 1 1 1 1 2 1 1 1 2 2 1 1 2 1 2 1 1 2 2 2 1 2 1 2

No 27 8 9 0 1 2=32

$f(1) =$ 1 2 2 2 2 2

$f(2) =$ 2 2 2 2 1 2

$f(3) =$ 2 2 2 1 2 2

$f(4) =$ 2 2 1 2 2 2

$f(5) =$ 2 1 2 2 2 2

$N = C_5^5(1;\theta) + C_5^4(1;2) + C_5^3(1;2,2) + C_5^2(1;2,2,2) + C_5^1(1;2,2,2,2) + C_5^5(2;\theta)$

N =1+5+10+10+5+1=32 positions=32 functions.

Application 4.2.3. $f: 1,2,3 \to 1,2,3$, N =27

No 1 2 3 4 5 6 7 8 9 0 1 2 3 4 5 6 7 8 9 0 1 2 3 4 5 6 7=27

$f(1) =$ 1 1 1 2 1 1 3 1 2 2 1 2 3 1 3 2 1 3 3 2 2 2 3 2 3 3 3

$f(2) =$ 1 1 2 1 1 3 1 2 2 1 2 3 1 3 2 1 3 3 1 2 2 3 2 3 3 2 3

$f(3) =$ 1 2 1 1 3 1 1 2 1 2 3 1 2 2 1 3 3 1 3 2 3 2 2 3 2 3 3

$N = C_3^3(1;\theta) + (C_3^2(1;2) + C_3^2(1;3)) + (C_3^1(1;2,2) + C_3^1(1;2,3) + C_3^1(1;3,2) + C_3^1(1;3,3)) + C_3^3(2;\theta) + C_3^2(2;3) + C_3^1(2;3,3) + C_3^3(3;\theta)$

N =1+(3+3)+(3+3+3+3)+1+3+3+1=27 positions=27 functions.

Application 4.2.4. $f: 1,2,3 \to 1,2,3,4$, N =64

No 1 2 3 4 5 6 7 8 9 0 1 2 3 4 5 6 7 8 9 0 1 2 3 4 5 6 7 8 9 0 1 2 3 4 5 6 7

$f(1) =$ 1 1 1 2 1 1 3 1 1 4 1 2 2 1 2 3 1 2 4 1 3 2 1 3 3 1 3 4 1 4 2 1 4 3 1 4 4

$f(2) =$ 11 2 1 1 3 1 1 4 1 2 2 1 2 3 1 2 4 1 3 2 1 3 3 1 3 4 1 4 2 1 4 3 1 4 4 1

$f(3) =$ 1 2 1 1 3 1 1 4 1 1 2 1 2 3 1 2 4 1 2 2 1 3 3 1 3 4 1 3 2 1 4 3 1 4 4 1 4

No 38 9 0 1 2 3 4 5 6 7 8 9 0 1 2 3 4 5 6 7 8 9 0 1 2 3 4=64

$f(1) =$ 2 2 2 3 2 2 4 2 3 3 2 3 4 2 4 3 2 4 4 3 3 3 4 3 4 4 4

$f(2) =$ 2 2 3 2 2 4 2 3 3 2 3 4 2 4 3 2 4 4 2 3 3 4 3 4 4 3 4

$f(3) =$ 2 3 2 2 4 2 2 3 2 3 4 2 3 3 2 4 4 2 4 3 4 3 3 4 3 4 4

$N = C_3^3(1;\theta)$ +[$C_3^2(1;2)$ + $C_3^2(1;3)$ + $C_3^2(1;4)$]+[$C_3^1(1;2,2)$ + $C_3^1(1;2,3)$ + $C_3^1(1;2,4)$ +

$+C_3^1(1;3,2)$ $+C_3^1(1;3,3)$ $+C_3^1(1;3,4)$ $+C_3^1(1;4,2)$ $+C_3^1(1;4,3)$ $+C_3^1(1;4,4)$]+

$+C_3^3(2;\theta)$ +[$C_3^2(2;3)$ $+C_3^2(2;4)$]+[$C_3^1(2;3,3)$ $+C_3^1(2;3,4)$ $+C_3^1(2;4,3)$ +

$+C_3^1(2;4,4)$]+ $C_3^3(3;\theta)$ $+C_3^2(3;4)$ $+C_3^1(3;4,4)$ $+C_3^3(4;\theta)$

There are a number of 10 groups of combinations.

N =1+[3+3+3]+[3+3+3+3+3+3+3+3+3]+1+[3+3]+[3+3+3+3]+1+3+3+1= =64 positions=64 functions. True.

Application 4.2.5. f : 1, 2,3,4 $\to$ 1,2,3, N =81

No 1 2 3 4 5 6 7 8 9 0 1 2 3 4 5 6 7 8 9 0 1 2 3 4 5 6 7 8 9 0 1 2 3 4 5 6 7=37

$f(1) =$ 1 1 1 1 2 1 1 1 3 1 1 2 2 1 2 1 1 2 3 1 2 1 1 3 2 1 3 1 1 1 3 3 1 3 1 2 2 2

$f(2) =$ 1 1 1 2 1 1 1 3 1 1 2 2 1 2 1 1 2 3 1 2 1 1 3 2 1 3 1 2 1 3 3 1 3 1 2 2 2 1

$f(3) =$ 1 1 2 1 1 1 3 1 1 2 2 1 1 1 2 2 3 1 1 1 3 3 2 1 1 1 2 1 3 3 1 1 1 3 2 2 1 2

$f(4) =$ 1 2 1 1 1 3 1 1 1 2 1 1 2 2 1 3 1 1 2 3 1 2 1 1 3 2 1 3 3 1 1 3 3 1 2 1 2 1 etc.

Now, for this case, we describe the decomposition of repetitive arrangements.

$N = C_4^4(1;\theta)$ +[$C_4^3(1;2)$ $+C_4^3(1;3)$]+[$C_4^2(1;2,2)$ $+C_4^2(1;2,3)$ $+C_4^2(1;3,2)$ + $C_4^2(1;3,3)$]+

+[$C_4^1(1;2,2,2)$ $+C_4^1(1;2,2,3)$ $+C_4^1(1;2,3,2)$ $+C_4^1(1;2,3,3)$ $+C_4^1(1;3,2,2)$ +

+ $C_4^1(1;3,2,3)$ + $C_4^1(1;3,3,2)$ + $C_4^1(1;3,3,3)$]+ $C_4^4(2;\theta)$ + $C_4^3(2;3)$ + $C_4^2(1;3,3)$ $+C_4^1(1;3,3,3)$ $+C_4^4(3;\theta)$.

There are a number of 9 groups of combinations.

N =1+[4+4]+[6+6+6+6+6]+[4+4+4+4+4+4+4+4]+1+4+6+4+1

N =1+[8]+[24]+[32]+1+4+6+4+1 = 81 positions=81 functions. True

5. General case for left-right algorithm and decomposition algorithm

The bijective functions are $f: A \to B$, $A = \left\{ a_1, a_2, \cdots, a_k \right\}$, $B = \left\{ b_1, b_2, \cdots, b_n \right\}$.

All combinations have the form C_k^i, $i = 1, k$; C_k^k, C_k^{k-1}, C_k^{k-2}, $\cdots, C_k^1$ =k; C_k^1 =0
and the function f is defined for $f(a_1)$, $f(a_2)$, $\ldots$, $f(a_k)$.

Remark 5.

$C_k^k(b_1; \theta)$, $k + card\{\theta\} = k$;

$C_k^{k-1}(b_1; x)$, for x: $b_2, b_3, \ldots, b_n$; $k - 1 + card\{x\} = k$;

$C_k^{k-2}(b_1; x, y)$; for x, y: $b_2, b_2; b_2, b_3$; *etc.*; $k - 2 + card\{x, y\} = k$;

$C_k^{k-3}(b_1; x, y, z)$, $k - 3 + card\{x, y, z\} = k$ etc.

We introduce some **adequate notations** and we analyze the algorithm for $k = 4$ and $n = 4$, $N = 256$. (**Version 3**).

Number m is used for combinations C_k^m, $m = k, k-1, \cdots, 3, 2, 1$.

j or b_j is the current number from $C_k^m(b_j; \cdots)$, where $m \le k$.

q indicates the number of completion elements taken from the set B in arrangements, with $m + q = k$.

p indicates the total elements from set $B = \left\{ b_1, b_2, \cdots, b_j, b_{j+1}, \cdots, b_n \right\}$ having the position in the right of b_j; $p = n - j$. We include the parameters p and q in the above notation and obtain $C_k^m(b_j; p; q)$. (1)

Verification: $p = n - j$, $m + q = k$ (2)

p^q is the partial number of repetitive arrangements for m, b_j, p, q (3)

$T(b_j; p; q)$ (natural number) is the last partial address generated by combinations

$C_k^m(b_j; p; q)$.

$T(b_j; p; q) = C_k^m \, p^q$ (4)

The sum of all $T(b_j; p; q)$ is N.

Application 5.1. Input data: $k=4$ and $n=4$, $N=256$; $f:$ 1,2,3,4 $\rightarrow$ 1,2,3,4.

For verification we use the formulas (1), (2), (3), (4).

$C_4^4(1;\theta)=1$, $T(1;\theta)=1$.

$C_4^3(1;p=3;q=1)=4$, $p^q=3$, $T(1;p=3;q=1)=C_4^3\,p^q$ =4x3=12 (product).

$p=3, q=1 \Rightarrow$ completion values are 2, 3, 4.

$C_4^2(1;p=3;q=2)=6$, $p^q=9$, $T(1;p=3;q=2)=C_4^2\,p^q$ =6 x 9=54.

$p=3, q=2 \Rightarrow$ completion values are 22, 23, 24;32,33,34;42,43,44.

$C_4^1(1;p=3;q=3)=4$, $p^q=27$, $T(1;3;q=3)=C_4^1\,p^q$ =4 x 27=108.

$p=3, q=3 \Rightarrow$ completion values are 222, 223, 224; 232, 233, 234; 242, 243,244 etc.

$C_4^4(2;\theta)=1$, $T(2;\theta)=1$.

$C_4^3(2;p=2;q=1)=4$, $p^q=2$, $T(2;p=2;q=1)=C_4^3\,p^q$ =4 x 2=8.

$p=2, q=1 \Rightarrow$ completion values are 3, 4.

$C_4^2(2;p=2;q=2)=6$, $p^q=4$, $T(2;p=2;q=2)=C_4^2\,p^q$ =6 x 4=24.

$p=2, q=2 \Rightarrow$ completion values are, 33, 34;43,44.

$C_4^1(2;p=2;q=3)=4$, $p^q=8$, $T(2;p=2;q=3)=C_4^1\,p^q$ =4 x 8=32.

$p=2, q=3 \Rightarrow$ completion values are 333, 334; 343, 344; 433,434; 443, 444.

$C_4^4(3;\theta)=1,$, $T(3;\theta)=1$.

$C_4^3(3;p=1;q=1)=4$, $p^q=1$, $T(3;p=1;q=1)=C_4^3\,p^q$ =4 x 1=4.

$p=1, q=1 \Rightarrow$ completion values are 4.

$C_4^2(3;p=1;q=2)=6$, $p^q=1$, $T(3;p=1;q=2)=C_4^2\,p^q$ =6 x 1=6.

$C_4^1(3;p=1;q=3)=4$, $p^q=1$, $T(3;p=1;q=3)=C_4^1\,p^q$ =4 x 1=4.

$p=1, q=3 \Rightarrow$ completion values are 444.

$C_4^4(4;\theta)=1$, $T(4;\theta)=1$.

Verification

$T(1;\theta)$ $+T(1;p=3;q=1)$ $+T(1;p=3;q=2)$ $+T(1;p=3;q=3)$ +

$+T(2;\theta)$ $+T(2;p=2;q=1)$ $+T(2;p=2;q=2)$ $+T(2;p=2;q=3)$ +

$+\ T(3;\theta)\ +\ T(3;p=1;q=1)\ +\ T(3;p=1;q=2)\ +\ T(3;p=1;q=3)\ +$
$+T(4;\theta)\ = N.$
$_{+}T(3;p=1;q=2)_{+}T(3;p=1;q=3)_{+}T(4;\theta)_{=N}$
1+12+54+108+1+8+24+32+1+4+6+4+1=256; N =256. True.

Application 5.2. f : 1, 2,3,4 $\rightarrow$ 1,2,3,4, N =256
We describe only the decomposition of repetitive arrangements.
There are a number of 60 groups of combinations.

N= $C_4^4(1;\theta)$ +[$C_4^3(1;2)$ + $C_4^3(1;3)$ + $C_4^3(1;4)$] + [$C_4^2(1;2,2)$ + $C_4^2(1;2,3)$ + $C_4^2(1;2,4)$ + $C_4^2(1;3,2)$ + $C_4^2(1;3,3)$ + $C_4^2(1;3,4)$ + $C_4^2(1;4,2)$ + $C_4^2(1;4,3)$ + $C_4^2(1;4,4)$]+ [$C_4^1(1;2,2,2)$ + $C_4^1(1;2,2,3)$ + $C_4^1(1;2,2,4)$ + $C_4^1(1;2,3,2)$ + $C_4^1(1;2,3,3)$ +
+ $C_4^1(1;2,3,4)$ + $C_4^1(1;2,4,2)$ + $C_4^1(1;2,4,3)$ + $C_4^1(1;2,4,4)$ + $C_4^1(1;3,2,2)$ +
+ $C_4^1(1;3,2,3)$ + $C_4^1(1;3,2,4)$ + $C_4^1(1;3,3,2)$ + $C_4^1(1;3,3,3)$ + $C_4^1(1;3,3,4)$ +
+ $C_4^1(1;3,4,2)$ + $C_4^1(1;3,4,3)$ + $C_4^1(1;3,4,4)$ + $C_4^1(1;4,2,2)$ + $C_4^1(1;4,2,3)$ +
+ $C_4^1(1;4,2,4)$ + $C_4^1(1;4,3,2)$ + $C_4^1(1;4,3,3)$ + $C_4^1(1;4,3,4)$ + $C_4^1(1;4,4,2)$ +
+ $C_4^1(1;4,4,3)$ + $C_4^1(1;4,4,4)$]+ $C_4^4(2;\theta)$ + [$C_4^3(2;3)$ + $C_4^3(2;4)$]+
+[$C_4^2(2;3,3)$ + $C_4^2(2;3,4)$ + $C_4^2(2;4,3)$ + $C_4^2(2;4,4)$]+
+[$C_4^1(2;3,3,3)$ + $C_4^1(2;3,3,4)$ + $C_4^1(2;3,4,3)$ + $C_4^1(2;3,3,4)$ + $C_4^1(2;4,3,3)$ +
+ $C_4^1(2;4,3,4)$ + $C_4^1(2;4,4,3)$ + $C_4^1(2;4,4,4)$]+ $C_4^4(3;\theta)$ + $C_4^3(3;4)$ +
+ $C_4^2(3;4,4)$ + $C_4^1(3;4,4,4)$ + $C_4^4(4;\theta)$.

N =1+(4x3)+(6x9)+(4x27)+1+(4x2)+(6x4)+(4x8)+1+4+6+4+1=256 positions, 256 functions.

6. The discrete functions applied to the practical problem

Application 6.1. Construct all electric circuits for 3 electric bulbs and 2 sources. The model is $f:\ \{a_1,a_2,a_3\}\ \rightarrow\ \{b_1,b_2\}$ or $f:\ \{1,2,3\}\ \rightarrow\ \{1,2\}$; N =8.

Solution. No 1 2 3 4 5 6 78 (circuit addresses)
$f(1)$ = 11 1 2 1 2 2 2
$f(2)$ = 1 1 2 1 2 2 1 2

$f(3) \;= 1\ 2\ 1\ 1\ 2\ 1\ 2\ 2$

The figure 4.1 shows all the circuits.

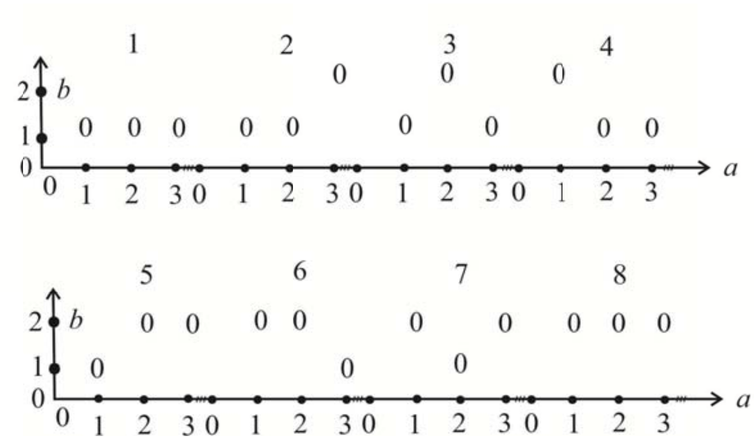

Figure 1. Application 6.1

Application 6.2. Construct all electric circuits for 2 electric bulbs and 3 sources. The model is $f:\ \{a_1, a_2\} \rightarrow \{b_1, b_2, b_3\}$ or $f:\ \{1,2\} \rightarrow \{1,2,3\}$; $N = 9$.

Solution. No 1 2 3 4 5 6 7 8 9 (circuit addresses)

$f(1) = 1\ 1\ 2\ 1\ 3\ 2\ 2\ 3\ 3$

$f(2) = 1\ 2\ 1\ 3\ 1\ 2\ 3\ 2\ 3$

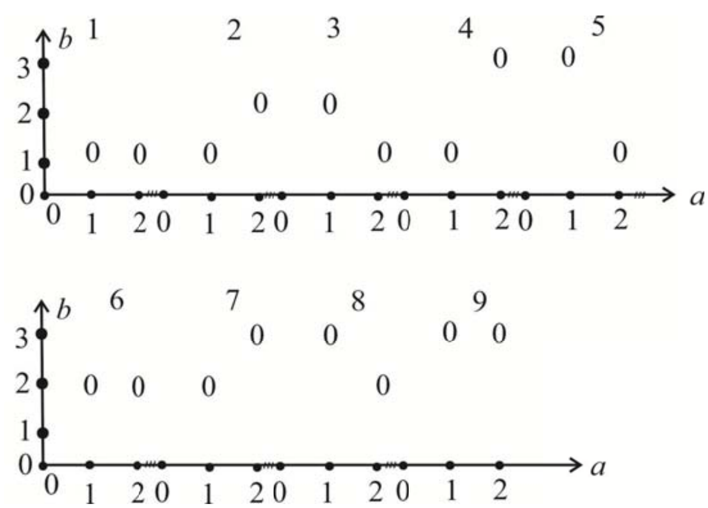

Figure 2. Application 6.2.

All other cases generates electrical circuits like the above circuits.

7. Bijective functions

We use $k = n$; $f: \{a_1, a_2, \cdots, a_n\} \to \{b_1, b_2, \cdots, b_n\}$ $n \geq 1$.
The total number of all bijective functions is $N_1 = n!$.

Application 7.1. f: 1,2,3 $\to$ 1,2,3; N_1 = 3! =6; $C_3^1(1,2,3)$ + $C_3^1(1,3,2)$ =3+3=6.

No 1 2 3 4 5 6 (addresses)
$f(1)$ = 12 3 1 3 2
$f(2)$ = 2 3 1 3 2 1
$f(3)$ = 3 12 2 1 3.

Application 7.2. f: 1,2,3,4 $\to$ 1,2,3,4; N_1 = 4! =24 (bijective functions).

$C_4^1(1,2,3,4)$ + $C_4^1(1,2,4,3)$ + $C_4^1(1,3,2,4)$ + $C_4^1(1,3,4,2)$ + $C_4^1(1,4,2,3)$ + $C_4^1(1,4,3,2)$ = + 4+4+4+4+4+4=24; C_4^1 =4.

No1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24
$f(1)$ = 1 2 3 4 1 2 4 3 1 3 2 4 1 3 4 2 1 4 2 3 1 4 3 2
$f(2)$ = 2 3 4 1 2 4 3 1 3 2 4 1 3 4 2 1 4 2 3 1 4 3 2 1
$f(3)$ = 3 4 1 2 4 3. 1 2 2 4 1 3 4 2 1 3 2 3 1 3 3 2 1 4
$f(4)$ = 4 1 2 3 3 1 2 4 4 1 3 2 2 1 3 4 3 1 4 2 2 1 4 3.

Application 7.3. f: 1,2,3,4,5 $\to$ 1,2,3,4,5; N_1 = 5! =120 (bijective functions); C_5^1 =5.

$C_5^1(1,2,3,4,5)$ + $C_5^1(1,2,3,5,4)$ + $C_5^1(1,2,4,3,5)$ + $C_5^1(1,2,4,5,3)$ + $C_5^1(1,2,5,3,4)$ + $C_5^1(1,2,5,4,3)$ **+ (** $C_5^1(1,3,2,4,5)$ + $C_5^1(1,3,2,5,4)$ + $C_5^1(1,3,4,2,5)$ + + $C_5^1(1,3,4,5,2)$ + $C_5^1(1,3,5,2,4)$ + $C_5^1(1,3,5,4,2)$) **+ (** $C_5^1(1,4,2,3,5)$ + $C_5^1(1,4,2,5,3)$ + $C_5^1(1,4,3,2,5)$ + $C_5^1(1,4,3,5,2)$ + $C_5^1(1,4,5,2,3)$ + $C_5^1(1,4,5,3,2)$) **+** $C_5^1(1,5,2,3,4)$ + $C_5^1(1,5,2,4,3)$ + $C_5^1(1,5,3,2,4)$ + $C_5^1(1,5,3,4,2)$ + **+**$C_5^1(1,5,4,2,3)$ + $C_5^1(1,5,4,3,2)$)=120.

Because the construction algorithm has direct access facility, we count all bijective functions, but we illustrate only some of them, related with

$$C_5^1(1,2,3,4,5) \text{ and } C_5^1(1,5,4,3,2).$$

No1 2 3 4 5 $\cdots\cdots$ 116 117118119120 (circuit addresses)

$f(1)$ = 1 2 3 4 5 $\cdots\cdots$ 154 3 2

$f(2)$ = 2 3 4 5 1 $\cdots\cdots$ 54 3 2 1

$f(3)$ = 3 4 51 2 $\cdots\cdots$ 432 1 5

$f(4)$ = 4 5 1 2 3 $\cdots\cdots$ 32 15 4

$f(5)$ = 5 1 2 3 4 $\cdots\cdots$ 2 1 5 4 3.

8. Conclusions

The work is presented at a level suitable for computer programming.

There are two methods to write a mathematical work. We begin by formulating a practical problem and to seek for the mathematical solution (direct problem). Or, we imagine a mathematical theory and look for a practical interpretation (inverse problem). Both methods are good. Our work is based on direct problem. But, now we can mention several new interpretations of discrete functions. For example, if we associate to each digit one color, we can apply the discrete functions in textile industry. Let us say f: 1,2,3 $\to$ 1 (red), 2 (green), 3 (yellow).

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