I^{-st} Section:

ECONOPHYSICS

INDUSTRIAL PROBLEMS OF TECHNICAL ELECTRODYNAMICS AND ANALYSIS OF THE INVERSE MATRIX OPERATOR EXISTENCE FOR THE "SYMMETRICAL" DIFFERENTIAL MAXWELL SYSTEM

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Abstract. *Quite a lot of industrial problems in technical electrodynamics are described by the so called "symmetrical" differential Maxwell system. This fact concerns the process of signal transmissions in the various kinds of media and analogous multidimensional circuits as well. The "symmetrical" feature means here the right-sided system completeness with respect to the first-ordered differential operator representation by the time argument. Though the computing technique is so broadly spread now, the analytic research tendencies of such industrial problems remain also required. Thus, the main goal of the present paper consists of the explicit mathematical study of the aforesaid Maxwell system by the combined analytic method. The latter differs from the well-known procedures but is in conformity with classical results and is simpler in practical application. The main stage of the proposed here method is the diagonalization process that reduces the symmetrical Maxwell system to the equivalent totality of scalar equations regarding the unknown components of the electromagnetic vector field functions. The given approach has the mixed character. Namely, the external diagonalization step "by blocks" transforms the original system to the uniform scalar – vector equation with respect to the electromagnetic vector field tensions. The next, inner diagonalization step "by coordinates" is replaced by more elegant and brief explicit construction of the corresponding inverse matrix differential operator. Conditions of such operator existence are studied by methods of applied mathematical analysis and are based on the original industrial problem statement.*

Keywords and phrases: *technical electrodynamics, "symmetrical" differential Maxwell system, diagonalization, inverse matrix operator, conditions of operator invertibility.*

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1. Introduction

Since the majority of physical and industrial problems can be described mathematically by the corresponding finite-dimensional system of partial differential equations (PDEs), its explicit analytical study remains urgently required [1]. This scientific direction concerns engineering aspects of technical electrodynamics as well, though the current computing technique is so broadly spread now [2].

Thus, the main purpose of present paper consists of the effective solution of some specific differential Maxwell system that represents the mathematical model of signal transmissions in the various kinds of media and multidimensional analogous circuits. An investigation approach is restricted here by the new diagonalization procedure that reduces the original system to the equivalent totality of respective scalar PDEs regarding the unknown components of the electromagnetic vector field tensions. Each PDE has only one unknown scalar function, and this fact allows solving the initial problem effectively.

The known preceding similar diagonalization technique was applied in direct manner using operator analogy of the algebraic Gauss method [3]-[6]. This algorithm acted through the whole matrix structure, involving all blocks and elements, either external or inner.

Though the mentioned method was uniform and quite satisfactory in simplicity of application, at some diagonalization stages it looked analytically rather tiresome. That was the main reason of birth of the suggested here combined diagonalization approach. It includes only the simpler external operator diagonalization step “by blocks” [3], [6]. Instead of its too long inner reiteration “by coordinates” [6], the appropriate inverse matrix operator is constructed.

Conditions of such operator existence base on the initial industrial problem statement and are studied completely for the symmetrical Maxwell system in the framework of technical electrodynamics.

2. Analytical problem statement

The so called “symmetrical” Maxwell system is written below:

$$\begin{cases} rot\,\vec{H} = (\sigma \pm \lambda\varepsilon_a)\vec{E} + \varepsilon_a \partial_0 \vec{E} + \overrightarrow{j^{OS}} \\ -rot\,\vec{E} = (r \pm \lambda\mu_a)\vec{H} + \mu_a \partial_0 \vec{H} + \overrightarrow{e^{OS}}. \end{cases} \tag{1}$$

In (1): the sought for vector functions $\vec{E},\vec{H} = \vec{E},\vec{H}(x,y,z,t)$ with scalar components $E_k, H_k = E_k, H_k(x,y,z,t)\ (k = \overline{1,3})$ represent electric and magnetic field tensions; the positive constants $\sigma, \mu_a, \varepsilon_a$ are the specific conductivity, absolute and dielectric permeability of the medium respectively; vector functions $\overrightarrow{j^{OS}}, \overrightarrow{e^{OS}} = \overrightarrow{j^{OS}}, \overrightarrow{e^{OS}}(x,y,z,t)$ are known and describe the outside current sources and tensions; $\lambda = const > 0$ is the parameter of the signal that intrudes into medium, and $r > 0$ is some theoretical constant whose existence can be only assumed at this research stage; (x,y,z,t) is the point of usual spatial Cartesian coordinate system; $\partial_0 = \frac{\partial}{\partial t}$; the classical rotor operator looks like:

$$rot = \det\begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_1 & \partial_2 & \partial_3 \\ F_{i1} & F_{i2} & F_{i3} \end{bmatrix}, \text{ where } \partial_1 = \frac{\partial}{\partial x}, \partial_2 = \frac{\partial}{\partial y}, \partial_3 = \frac{\partial}{\partial z},$$

and functions $F_{ik} = F_{ik}(x,y,z,t)\ (i = 1,2;\ k = \overline{1,3})$ are scalar components of the appropriate vector functions $\vec{F_1}, \vec{F_2} = \vec{F_1}, \vec{F_2}(x,y,z,t)$; $\vec{F_1}, \vec{F_2} = \vec{E}, \vec{H}$.

"Symmetry" means here the right-sided system completeness with respect to the first-ordered differential operator representation by the time variable t.

Interchange of sign in front of λ means the response of medium to signal excitation. Symbol of plus means the absorption of signal by medium and minus – the seizure of signal.

Formula (1) is the mathematical model of the multidimensional analogous circuits and signal transmissions in the various kinds of media. Both phenomena concern the most important parts of modern technical electrodynamics.

As it was said above, effective analytical study of system (1) is done in terms of new combined diagonalization method that reduces the original problem to the equivalent totality of corresponding scalar PDEs regarding the unknown components of vector functions $\vec{E}, \vec{H}$. Each PDE depends on the only one element of such type. The last fact immediately leads to the explicit solution of the original problem.

3. Results

The aforesaid mixed diagonalization process of system (1) consists of three stages.

The first one is the known simple operator diagonalization procedure "by blocks" that reduces (1) to the uniform scalar – vector equation with respect to $\vec{E}, \vec{H}$ [3], [6]:

$$-(A^2 + DC)\vec{F_i} = \vec{\varphi_i} \quad (i = 1, 2), \tag{2}$$

where:

$$\vec{F_1} = \vec{E},\ \vec{F_2} = \vec{H};\ \vec{\varphi_1} = A\overrightarrow{e^{OS}} + D\overrightarrow{j^{OS}}, \vec{\varphi_2} = C\overrightarrow{e^{OS}} - A\overrightarrow{j^{OS}};$$

$$A = rot, D = r + \mu_a \partial_0^*,\ C = \sigma + \varepsilon_a \partial_0^*,\ \partial_0^* = \partial_0 \pm \lambda. \tag{3}$$

Equation (2), (3) was obtained in [3], [6].

Now it should be noted that operator product everywhere in the given paper means usual operator consecutive application from the inner to the external.

The second stage deals with the explicit construction of the inverse differential matrix operator. This approach is proposed instead of the cumbersome diagonalization of (2), (3) "by coordinates" (look [6]). Only reduction of (2), (3) to the equivalent non diagonalized system of scalar PDEs from [6] was used:

$$\begin{cases} A_{23}F_{i1} - B_{12}F_{i2} - B_{13}F_{i3} = \varphi_{i1} \\ -B_{12}F_{i1} + A_{13}F_{i2} - B_{23}F_{i3} = \varphi_{i2} \quad (i = 1,2). \\ -B_{13}F_{i1} - B_{23}F_{i2} + A_{12}F_{i3} = \varphi_{i3} \end{cases} \tag{4}$$

In (4): F_{ik}, φ_{ik} $(i = 1,2; k = \overline{1,3})$ are scalar components of the unknown and given vector functions $\vec{F_i}, \vec{\varphi_i}$ $(i = 1,2)$ from (3) respectively; operator matrix elements look like:

$$\Delta = \sum_{k=1}^{3} \partial_k^2;\ \ \hat{\partial}_0^2 = \mu_a \varepsilon_a (\partial_0^*)^2 + (\sigma\mu_a + r\varepsilon_a)\partial_0^* + r\sigma;$$

$$B_{jk} = \partial_j \partial_k\ (j \neq k);\ A_{jk} = \Delta - \partial_l^2 - \hat{\partial}_0^2\ (l \neq j,k;\ j \neq k)\ \ (j,k,l = \overline{1,3}). \tag{5}$$

Other notation in (5) is from (1), (3).

Further, the system (4) is rewritten in matrix form

$$KF_i = \varphi_i \quad (i = 1,2), \tag{6}$$

Where:

$$K = \begin{bmatrix} A_{23} & -B_{12} & -B_{13} \\ -B_{12} & A_{13} & -B_{23} \\ -B_{13} & -B_{23} & A_{12} \end{bmatrix}, \quad F_i = \begin{bmatrix} F_{i1} \\ F_{i2} \\ F_{i3} \end{bmatrix}, \quad \varphi_i = \begin{bmatrix} \varphi_{i1} \\ \varphi_{i2} \\ \varphi_{i3} \end{bmatrix} \; (i = 1,2). \tag{7}$$

Taking into account operator analogy of the classical matrix theory [7], the inverse matrix operator for K is constructed explicitly.

It is easy to notice that the original matrix operator from (7) is symmetrical with respect to the main diagonal. Hence, the sought for inverse operator K^{-1} must possess the same property [7]. Really:

$$K^{-1} = (\det K)^{-1} \begin{bmatrix} K_{11} & K_{21} & K_{31} \\ K_{12} & K_{22} & K_{32} \\ K_{13} & K_{23} & K_{33} \end{bmatrix}, \tag{8}$$

and $K_{mn}\ (m,n = \overline{1,3})$ represent operator analogies of the corresponding algebraic supplements from K.

In (8):

$$\det K = -\hat{\partial}_0^2(\hat{\partial}_0^2 - \Delta)^2; \; K_{mm} = (\Delta - \hat{\partial}_0^2)(\partial_m^2 - \hat{\partial}_0^2) \; (m = \overrightarrow{1,3});$$

$$K_{mn} = K_{nm} = \partial_m \partial_n (\Delta - \hat{\partial}_0^2) \; (m,n = \overrightarrow{1,3};\, m \neq n) \tag{9}$$

and $\det$ is the usual determinant [7] in the operator meaning.

The substitution of (9) for (8) that is the explicit construction for K^{-1} is obtained after obvious algebraic transformation. Namely,

$$K^{-1} = (\hat{\partial}_0^2)^{-1}(\hat{\partial}_0^2 - \Delta)^{-1} \begin{bmatrix} \partial_1 - \hat{\partial}_0^2 & \partial_1\partial_2 & \partial_1\partial_3 \\ \partial_1\partial_2 & \partial_2 - \hat{\partial}_0^2 & \partial_2\partial_3 \\ \partial_1\partial_3 & \partial_2\partial_3 & \partial_3 - \hat{\partial}_0^2 \end{bmatrix}. \tag{10}$$

Direct simple verification confirms $K^{-1}K = KK^{-1} = I = diag\,(1,1,1)$ and agrees with the main property of inverse matrix operator [7].

Then operator K^{-1} from (10) is applied to the both parts of the matrix equation (6), (7) that leads to the required equivalent totality of ordinary scalar PDEs:

$$F_i = \begin{bmatrix} F_{i1} \\ F_{i2} \\ F_{i3} \end{bmatrix} = K^{-1}\varphi_i = K^{-1}\begin{bmatrix} \varphi_{i1} \\ \varphi_{i2} \\ \varphi_{i3} \end{bmatrix} \Leftrightarrow \begin{bmatrix} F_{i1} \\ F_{i2} \\ F_{i3} \end{bmatrix} =$$

$$= (\hat{\partial}_0^2(\hat{\partial}_0^2 - \Delta))^{-1} \begin{bmatrix} \partial_1^2 - \hat{\partial}_0^2 & \partial_1\partial_2 & \partial_1\partial_3 \\ \partial_1\partial_2 & \partial_2^2 - \hat{\partial}_0^2 & \partial_2\partial_3 \\ \partial_1\partial_3 & \partial_2\partial_3 & \partial_3^2 - \hat{\partial}_0^2 \end{bmatrix} \begin{bmatrix} \varphi_{i1} \\ \varphi_{i2} \\ \varphi_{i3} \end{bmatrix} \; (i = 1,2). \qquad (11)$$

It is easy to check that the final result (11) is identical to the sought for system from paper [6]. Besides, the suggested current diagonalization method is mathematically more elegant and thrice-shorter, at least:

$$\hat{\partial}_0^2(\hat{\partial}_0^2 - \Delta) \begin{bmatrix} F_{i1} \\ F_{i2} \\ F_{i3} \end{bmatrix} = \begin{bmatrix} (\partial_1^2 - \hat{\partial}_0^2)\varphi_{i1} + \partial_1(\partial_2\varphi_{i2} + \partial_3\varphi_{i3}) \\ (\partial_2^2 - \hat{\partial}_0^2)\varphi_{i2} + \partial_2(\partial_1\varphi_{i1} + \partial_3\varphi_{i3}) \\ (\partial_3^2 - \hat{\partial}_0^2)\varphi_{i3} + \partial_3(\partial_1\varphi_{i1} + \partial_2\varphi_{i2}) \end{bmatrix} (i = 1,2). \qquad (12)$$

Hence, comparing the obtained formula (11) and required result of [6] that is reflected in (12), it is easy to find their complete identity.

The given analytic technique was briefly announced in [8].

It seems from the first sight now that the original problem is solved and the purpose of given paper is attained. Also, it is not so, since formula (11) includes the expression:

$$(\hat{\partial}_0^2(\hat{\partial}_0^2 - \Delta))^{-1}. \qquad (13)$$

The latter is necessary in the explicit construction of the inverse matrix operator K^{-1} from (10). Therefore, it is clear, when (13) in general does not exist the suggested diagonalization method can not be accepted as non valid.

This problem raises the third closing stage of present investigation that concerns the existence of inverse operator (13) with respect to:

$$\hat{\partial}_0^2(\hat{\partial}_0^2 - \Delta). \qquad (14)$$

Since $\hat{\partial}_0^2$ is the particular case of $\hat{\partial}_0^2 - \Delta$ when $\Delta = 0$, complete study in the case of $\hat{\partial}_0^2 - \Delta$ is quite satisfactory and enough.

In other words, existence of the inverse operator (13) implies execution of the following condition [9]:

$$Ker\ (\hat{\partial}_0^2 - \Delta) = \varnothing. \qquad (15)$$

Here Ker is the operator kernel and describes those respective sets of functions which are transformed in zero after the given operator application [9].

When condition (15) is not true, it means existence of the nonempty functional set whose elements satisfy the equality:

$$(\hat{\partial}_0^2 - \Delta) f = 0,\ f = f(x, y, z, t). \qquad (16)$$

Hence, if such functional set from (16) is found, it must be excluded from those functions which guarantee the validity of (15). Therefore, the appropriate operator equation must be considered:

$$(\mu_a \varepsilon_a (\partial_0^*)^2 + (\sigma\mu_a + r\varepsilon_a)\partial_0^* + (r\sigma - \Delta)) f = 0, \qquad (17)$$

and it represents the preceding equation (16) that is rewritten in more suitable form. In its turn, formally (17) can be reduced to simpler analogy of quadratic with respect to operator ∂_0^*:

$$\mu_a \varepsilon_a (\partial_0^*)^2 + (\sigma\mu_a + r\varepsilon_a)\partial_0^* + (r\sigma - \Delta) = 0. \qquad (18)$$

Solving (18) in usual elementary algebraic manner find its discriminant:

$$| D = (\sigma\mu_a + r\varepsilon_a)^2 - 4\mu_a\varepsilon_a(r\sigma - \Delta) = (\sigma\mu_a - r\varepsilon_a)^2 + 4\mu_a\varepsilon_a\Delta \qquad (19)$$

whose first item and constant factor near the Laplace operator in the second item are positive. Since values of Δ after its application to functions $f = f(x, y, z, t)$ are arbitrary real, then values of (19) when applying to the same objects are arbitrary real too.

The initial industrial problem and its mathematical model (1) are considered in four–dimensional real space (x, y, z, t). Since functions f are real-valued, then in the case of (19) being negative, equivalent equations (16)-(18) have the empty set of solutions and condition (15) is true. It means, in its turn, that the inverse operator (10) really exists for all possible f which are finitely derivated in some set from the space $\{(x,y,z,t)\}$. Mathematically this condition of operator existence looks as:

$$D<0 \Leftrightarrow (\sigma\mu_a - r\varepsilon_a)^2 + 4\mu_a\varepsilon_a\Delta < 0 \Leftrightarrow \Delta < -(\sigma\mu_a - r\varepsilon_a)^2/(4\mu_a\varepsilon_a) < 0 \Leftrightarrow$$

$$\Leftrightarrow \Delta < -\frac{1}{4}\left(\sigma\sqrt{\frac{\mu_a}{\varepsilon_a}} - r\sqrt{\frac{\varepsilon_a}{\mu_a}}\right)^2 < 0 \qquad (20)$$

and the first condition of existence of the inverse operator (10) is obtained.

Further, let $D \geq 0$ in (19). Then solution of (18) is the following:

$$\partial_0^* = \frac{-(\sigma\mu_a + r\varepsilon_a) \pm \sqrt{D}}{2\mu_a\varepsilon_a}. \qquad (21)$$

Taking into account formulae (19) and (3), expression (21) can be reduced to the equivalent one:

$$\left|\frac{\partial}{\partial t} = \mp\lambda - \frac{(\sigma\mu_a + r\varepsilon_a) \mp \sqrt{(\sigma\mu_a - r\varepsilon_a)^2 + 4\mu_a\varepsilon_a\Delta}}{2\mu_a\varepsilon_a}\right.. \qquad (22)$$

In (22), the signs in front of λ and $\sqrt{\ }$ change independently of each other, i.e. there are four possible versions in reality. Namely, "– , –"; "– , +"; "+, –"; "+, +", where the first sign in the given pairs corresponds with λ and the second conforms to $\sqrt{\ }$.

Thus, according to the last remark, condition (22) can be expressed as follows:

$$\frac{\partial}{\partial t} = -\frac{1}{2}\left(\left(\frac{\sigma}{\varepsilon_a} + \frac{r}{\mu_a}\right) \mp \sqrt{\left(\frac{\sigma}{\varepsilon_a} - \frac{r}{\mu_a}\right)^2 + \frac{4\Delta}{\mu_a\varepsilon_a}}\right) + \begin{bmatrix} -\lambda \\ +\lambda \end{bmatrix}. \qquad (23)$$

Basing on (23) and on the condition of $D \geq 0$ in (20), the next restriction of the inverse operator existence in terms of (15) is written below:

$$\begin{cases} \Delta \geq -\left(\frac{1}{2}\left(\sigma\sqrt{\frac{\mu_a}{\varepsilon_a}} - r\sqrt{\frac{\varepsilon_a}{\mu_a}}\right)\right)^2 \\ \partial_0 = \frac{\partial}{\partial t} \neq \begin{bmatrix} -\lambda \\ +\lambda \end{bmatrix} - \frac{1}{2}\left(\left(\frac{\sigma}{\varepsilon_a} + \frac{r}{\mu_a}\right) \mp \sqrt{\left(\frac{\sigma}{\varepsilon_a} - \frac{r}{\mu_a}\right)^2 + \frac{4\Delta}{\mu_a\varepsilon_a}}\right) \end{cases}. \qquad (24)$$

The second condition in (24) means that values of the differential operator after its application by the time argument t must not coincide with quantities from the right side of this inequality.

Gathering all obtained results concerning the required inverse operator existence, the final condition can be represented as the union of (20) and (24):

$$\left[\begin{array}{l} \Delta < -\left(\frac{1}{2}\left(\sigma\sqrt{\frac{\mu_a}{\varepsilon_a}} - r\sqrt{\frac{\varepsilon_a}{\mu_a}} \right) \right)^2 \\ \left\{ \begin{array}{l} \Delta \geq -\left(\frac{1}{2}\left(\sigma\sqrt{\frac{\mu_a}{\varepsilon_a}} - r\sqrt{\frac{\varepsilon_a}{\mu_a}} \right) \right)^2 \\ \partial_0 = \frac{\partial}{\partial t} \neq \begin{bmatrix} -\lambda \\ +\lambda \end{bmatrix} - \frac{1}{2}\left(\left(\frac{\sigma}{\varepsilon_a} + \frac{r}{\mu_a} \right) \mp \sqrt{\left(\frac{\sigma}{\varepsilon_a} - \frac{r}{\mu_a} \right)^2 + \frac{4\Delta}{\mu_a \varepsilon_a}} \right) \end{array} \right. \end{array} \right. . \quad (25)$$

Formula (25) completes the last third stage of the original problem study. Hence, the initially raised problem is solved effectively and the purpose of given paper is attained.

4. Remarks and conclusions

Though the proposed here method looks more analytical than the preceding one that was reflected in [3]-[6], it implies only the same knowledge of linear algebra, since condition (15) remains for square matrices too. So, this given technique is at the same level of understanding and simplicity for engineers as the known one from [3]-[6]. Moreover, the suggested present approach requires only preceding operator restrictions which were accepted earlier in [3]-[6]. They consist of operator commutativity in pairs and their invertibility. The explicit scalar inverse operator construction (of (13), for example) is not required at all, only operator invertibility is assumed.

Turning again to [3]-[6], it is easy to find that the mentioned property of operator being invertible was accepted a'priori there, though in ideal it had to be checked for all possible cases that were met in the specific considered problem. The current diagonalization method validity is checked for one general stage of study, at least. Besides, this diagonalization step "by coordinates" in terms of inverse matrix operator, is mostly important because of too long, dull and cumbersome direct calculation at the similar stage of investigation in [3]-[6].

Moreover, the suggested analytic approach is valid for the same class of industrial problems that was considered in [3]-[6], and does not imply any additional restrictions of its application in comparison with earlier known results.

Finally, the present procedure is far shorter than the aforesaid one from [3]-[6] and appears clear and strict mathematically, remaining simultaneously easy for engineering applications.

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